Math 4305, Exam 2: 3.2-5.4 (practice) Name/Section: ____________________

1. (25 points) (3.7.6) Find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{pmatrix}.$$

Solution: We apply the row reduction algorithm to obtain

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \end{pmatrix}.$$

From this we see there are two pivots and the rank is 2.

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2. (25 points) (4.3.20) Let $\mathcal{P}$ be the collection of quadratic (and lower order) polynomials in t. Let $\mathcal{B}$ denote the basis $\{1, t, t^2\}$ for $\mathcal{P}$. Find the matrix corresponding to the linear transformation $T : \mathcal{P} \to \mathcal{P}$ given by $T(f) = f'$ with respect to the basis $\mathcal{B}$.

Solution: We simply take the images of the basis vectors $T(1) = 0$, $T(t) = 1$, $T(t^2) = 2t$ and express each one as a linear combination of the basis vectors:

$$T(1) = 0(1) + 0(t) + 0(t^2), \quad T(t) = 1(1) + 0(t) + 0(t^2), \quad T(t^2) = 0(1) + 2(t) + 0(t^2),$$

and put the coefficients in the columns:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

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3. (25 points) (5.3.40) Find the orthogonal projection of $\mathbb{R}^4$ onto

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ -5 \\ 3 \end{pmatrix} \right\}.$$

Solution: First we apply the Gram-Schmidt process to get an orthonormal basis for the subspace:

$$\mathbf{v}_1 = (1,1,1,1)^T/2$$

$$\mathbf{w}_2 = (1,9,-5,3)^T - (1,9,-5,3)^T \cdot \mathbf{v}_1 \mathbf{v}_1 = (-1,7,-7,1)^T.$$

$$\mathbf{v}_2 = (-1,7,-7,1)^T/10.$$

Next, we put these orthonormal vectors in the columns of a matrix and left multiply it on its transpose to get the projection:

$$P = \frac{1}{100} \begin{pmatrix} 5 & -1 \\ 5 & 7 \\ 5 & -7 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 5 & 5 & 5 & 5 \\ -1 & 7 & -7 & 1 \end{pmatrix} = \frac{1}{100} \begin{pmatrix} 26 & 18 & 32 & 24 \\ 18 & 74 & -24 & 32 \\ 32 & -24 & 74 & 18 \\ 24 & 32 & 18 & 26 \end{pmatrix}$$

$$= \frac{1}{50} \begin{pmatrix} 13 & 9 & 16 & 12 \\ 9 & 37 & -12 & 16 \\ 16 & -12 & 37 & 9 \\ 12 & 16 & 9 & 13 \end{pmatrix}.$$

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4. (25 points) (5.4.23) Find the least-squares approximation for

$$\begin{pmatrix} 1 & 1 \\ 2 & 8 \\ 1 & 5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

Solution: Consider instead the solvable equation

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 8 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 8 \\ 1 & 5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 8 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

obtained by multiplying both sides of the original equation by the transpose of the matrix on the left. That is,

$$\begin{pmatrix} 6 & 22 \\ 22 & 90 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the matrix

$$\begin{pmatrix} 6 & 22 \\ 22 & 90 \end{pmatrix} = 2 \begin{pmatrix} 3 & 11 \\ 11 & 45 \end{pmatrix}$$

represents a nonsingular (one-to-one and onto) mapping of $\mathbb{R}^2$ onto itself, the unique solution of this equation is $\mathbf{x} = (0, 0)^T$. And this is the closest you can get to a solution of the original equation.