Math 4305, Exam 1: 1.1-3.1 (practice) Name/Section: ____________

1. (25 points) (1.2.6) Find the general solution of the system of equations

$$\begin{cases} x_1 - 7x_2 + x_5 = 3 \\ x_3 - 2x_5 = 2 \\ x_4 + x_5 = 1. \end{cases}$$

Solution: We apply the row reduction algorithm to the augmented coefficient matrix. We see that the matrix is already in reduced echelon form with pivots in columns 1, 3, and 4.

$$\begin{pmatrix} 1 & -7 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Thus, x_2 and x_5 are free and the solution set has two parameters

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 7 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}.$$

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2. (25 points) (2.3.7) Find the product AB if

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{pmatrix}.$$

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3. (25 points) (2.4.6) Find the inverse matrix A^{-1} if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution: We append the identity matrix and row reduce:

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, we see the inverse is given by

$$A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

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4. (25 points) (3.1.22) Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(\mathbf{x}) = \begin{pmatrix} 2x_1 + x_2 + 3x_3 \\ 3x_1 + 4x_2 + 2x_3 \\ 6x_1 + 5x_2 + 7x_3 \end{pmatrix}.$$

What is the image of T?

Solution: Let us begin by testing the hypothesis that T is invertible (and the image is all of $\mathbb{R}^3$).

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 3 \\ 0 & 4 - 3/2 & 2 - 9/2 \\ 0 & 2 & -2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 2 & 1 & 3 \\ 0 & 5 & -5 \\ 0 & 2 & -2 \end{pmatrix}.$$

This means the matrix is not invertible. There are pivots, however, in the first and second columns. This means the first and second columns of the original matrix span the image.

$$\mathrm{Im}(T) = \mathrm{span}\left\{ \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} \right\}$$

is a 2-dimensional plane in $\mathbb{R}^3$.