Challenge Problem

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Question:

W is a subspace of $\mathbf{R}^n$ with basis **B** = $\{w_1, w_2, \ldots, w_k\}$. Let $v \in W$ be expressed in the standard basis. What is $[v]_{\mathbf{B}}$?

Solution:

Let us represent v as the column vector:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{pmatrix}$$

This can be rewritten as:

$$v = v_1\hat{e}_1 + v_2\hat{e}_2 + \cdots + \ v_n\hat{e}_n$$

Now, we are required to find $[v]_{\mathbf{B}}$ which denotes the coordinates of v expressed in the basis **B**. To find this, we have to solve the following system:

$$v = c_1w_1 + c_2w_2 + \cdots + c_kw_k$$

Here $c_1, c_2, \ldots, c_k$ are the coordinates. The system is to be solved for these coordinates.

Recasting the above equation in matrix form, we get:

$$\begin{pmatrix} | & | & . & . & . & | \\ w_1 & w_2 & . & . & . & w_k \\ | & | & . & . & . & | \end{pmatrix}_{n*k} * \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_k \end{pmatrix}_{k*1} = \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{pmatrix}_{n*1}$$

Let us make the following notation to write the above concisely:

$$A = \begin{pmatrix} | & | & . & . & . & | \\ w_1 & w_2 & . & . & . & w_k \\ | & | & . & . & . & | \end{pmatrix}_{n*k} \quad ; \qquad c = \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_k \end{pmatrix}_{k*1}$$

Using this notation, our equation becomes:

$$Ac = v : A \text{ system of } n \text{ equations in } k \text{ unknowns}$$

Let us think of this system in terms of a linear transformation:

$$T: R^k \to R^n ; T(x) = Ax$$

Now, for the above system to have a solution, v should lie in the column space of A. The column space of A is the image space of T. This image space is generated by the span of the columns of A. But, we know that:

$$W = span(w_1, w_2, \dots, w_k)$$

Thus, for the system to have a solution, we require v to lie in the subspace **W**. However, suppose that **W** is the xy plane in $\mathbf{R}^3$. Any vector with a non-zero component along the z axis, will not belong to **W**. Thus, to express such a vector in the coordinates with respect to the basis **B** of the subspace **W**, we will have to project the vector onto the subspace **W**.

Hence, we have to actually solve the modified system:

$$Ac = proj_{col(A)} v$$

In terms of a linear transformation, we have to find a vector c in $\mathbf{R}^k$ which, under the transformation T, maps to the projection of v onto the column space of A (which is the subspace **W** in $\mathbf{R}^n$). Now, since the matrix A is non-square and hence non-invertible, we use the transpose of A to map in the opposite direction viz. from $\mathbf{R}^n$ to $\mathbf{R}^k$.

Now, we know that:

$$v = proj_{col(A)} v + perp_{col(A)} v$$

To map v back to **R**^{k}, we operate the transpose of A on V:

$$A^T v = A^T(proj_{col(A)}v + perp_{col(A)}v)$$

But,

$$A^T perp_{col(A)}v = 0$$; The transpose kills the orthogonal complement

Thus,

$$A^T v = A^T proj_{col(A)}v$$

Now, recall that we have to solve the modified system:

$$Ac = proj_{col(A)}v$$

$$\Rightarrow A^T Ac = A^T proj_{col(A)}v$$

$$\Rightarrow c = (A^T A)^{-1} A^T proj_{col(A)}v$$

But, we saw that the transpose operator gives the same result operated on v or its projection. Thus:

$$c = (A^T A)^{-1} A^T v$$