EXAMPLE 9 Let V be the space of all infinite sequences of real numbers. We define the linear transformation

$$T(x_0, x_1, x_2, x_3, x_4, \ldots) = (x_1, x_2, x_3, x_4, \ldots)$$

from V to V (we omit the first term, x_0). Find all the eigenvalues and eigensequences of T.

Solution

Since V is infinite dimensional, we cannot use the matrix techniques of Example 8 here. We have to go back to the definition of an eigenvalue: For a fixed scalar λ, we are looking for the infinite sequences $(x_0, x_1, x_2, x_3, \ldots)$ such that

$$T(x_0, x_1, x_2, x_3, \ldots) = \lambda(x_0, x_1, x_2, x_3, \ldots)$$

or

$$(x_1, x_2, x_3, \ldots) = \lambda(x_0, x_1, x_2, x_3, \ldots)$$

or

$$x_1 = \lambda x_0, \quad x_2 = \lambda x_1 = \lambda^2 x_0, \quad x_3 = \lambda x_2 = \lambda^3 x_0, \ldots.$$

The solutions are the *geometric sequences* of the form

$$(x_0, \lambda x_0, \lambda^2 x_0, \lambda^3 x_0, \ldots) = x_0(1, \lambda, \lambda^2, \lambda^3, \ldots).$$

Thus all real numbers λ are eigenvalues of T, and the eigenspace E_λ is one dimensional for all λ, with the geometric sequence $(1, \lambda, \lambda^2, \lambda^3, \ldots)$ as a basis.

For example, when $\lambda = 3$, we have

$$T(1, 3, 9, 27, 81, \ldots) = (3, 9, 27, 81, \ldots) = 3(1, 3, 9, 27, \ldots),$$

demonstrating that $(1, 3, 9, 27, 81, \ldots)$ is an eigensequence of T with eigenvalue 3. ∎

EXERCISES 7.4

GOAL *Use the concept of a diagonalizable matrix. Find the eigenvalues of a linear transformation.*

Decide which of the matrices A in Exercises 1 through 20 are diagonalizable. If possible, find an invertible S and a diagonal D such that $S^{-1}AS = D$. Do not use technology.

1. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
2. $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$
3. $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$
4. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$
5. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
6. $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$
7. $\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$
8. $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$
9. $A = \begin{bmatrix} 4 & 9 \\ -1 & -2 \end{bmatrix}$
10. $A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$
11. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
12. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
13. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$
14. $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
15. $A = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
16. $A = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
17. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
18. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
19. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
20. $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$