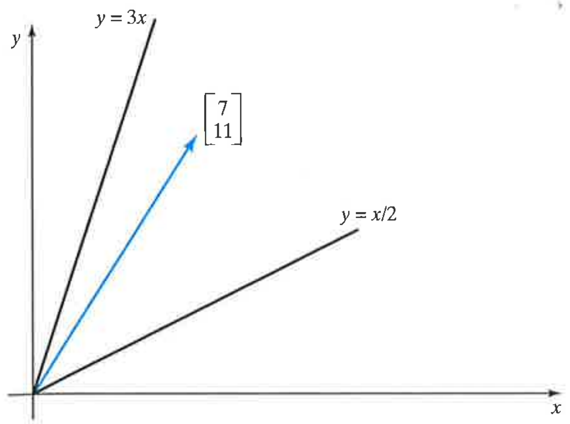
57. Express the vector $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$ as the sum of a vector on the line $y = 3x$ and a vector on the line $y = x/2$.



58. For which values of the constants b and c is the vector $\begin{bmatrix} 3 \\ b \\ c \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}$?

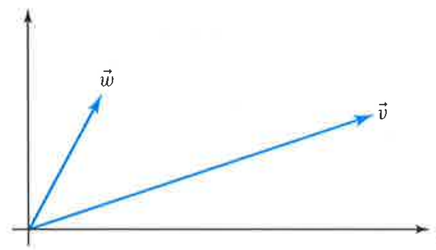
59. For which values of the constants c and d is $\begin{bmatrix} 5 \\ 7 \\ c \\ d \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$?

60. For which values of the constants a, b, c, and d is $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ a linear combination of $\begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix}$?

61. For which values of the constant c is $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$?

62. For which values of the constant c is $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}$, where a and b are arbitrary constants?

In Exercises 63 through 67, consider the vectors $\vec{v}$ and $\vec{w}$ in the accompanying figure.



63. Give a geometrical description of the set of all vectors of the form $\vec{v} + c\vec{w}$, where c is an arbitrary real number.

64. Give a geometrical description of the set of all vectors of the form $\vec{v} + c\vec{w}$, where $0 \le c \le 1$.

65. Give a geometrical description of the set of all vectors of the form $a\vec{v} + b\vec{w}$, where $0 \le a \le 1$ and $0 \le b \le 1$.

66. Give a geometrical description of the set of all vectors of the form $a\vec{v} + b\vec{w}$, where $a + b = 1$.

67. Give a geometrical description of the set of all vectors of the form $a\vec{v} + b\vec{w}$, where $0 \le a$, $0 \le b$, and $a + b \le 1$.

68. Give a geometrical description of the set of all vectors $\vec{u}$ in $\mathbb{R}^2$ such that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$.

69. Solve the linear system

$$\left| \begin{aligned} y + z &= a \\ x \quad + z &= b \\ x + y \quad &= c \end{aligned} \right|,$$

where a, b, and c are arbitrary constants.

70. Let A be the $n \times n$ matrix with 0's on the main diagonal, and 1's everywhere else. For an arbitrary vector $\vec{b}$ in $\mathbb{R}^n$, solve the linear system $A\vec{x} = \vec{b}$, expressing the components $x_1, \dots, x_n$ of $\vec{x}$ in terms of the components of $\vec{b}$. (See Exercise 69 for the case $n = 3$.)

b. A homogeneous system with fewer equations than unknowns has infinitely many solutions.

c. If $\vec{x}_1$ and $\vec{x}_2$ are solutions of the homogeneous system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_2$ is a solution as well.

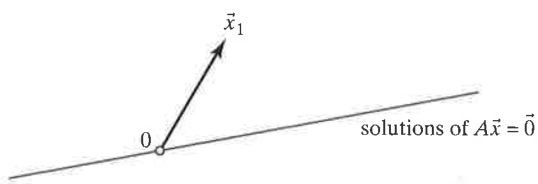
d. If $\vec{x}$ is a solution of the homogeneous system $A\vec{x} = \vec{0}$ and k is an arbitrary constant, then $k\vec{x}$ is a solution as well.

48. Consider a solution $\vec{x}_1$ of the linear system $A\vec{x} = \vec{b}$. Justify the facts stated in parts (a) and (b):

a. If $\vec{x}_h$ is a solution of the system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_h$ is a solution of the system $A\vec{x} = \vec{b}$.

b. If $\vec{x}_2$ is another solution of the system $A\vec{x} = \vec{b}$, then $\vec{x}_2 - \vec{x}_1$ is a solution of the system $A\vec{x} = \vec{0}$.

c. Now suppose A is a 2×2 matrix. A solution vector $\vec{x}_1$ of the system $A\vec{x} = \vec{b}$ is shown in the accompanying figure. We are told that the solutions of the system $A\vec{x} = \vec{0}$ form the line shown in the sketch. Draw the line consisting of all solutions of the system $A\vec{x} = \vec{b}$.



If you are puzzled by the generality of this problem, think about an example first:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \text{and} \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

49. Consider the accompanying table. For some linear systems $A\vec{x} = \vec{b}$, you are given either the rank of the coefficient matrix A, or the rank of the augmented matrix $\left[A \mid \vec{b}\right]$. In each case, state whether the system could have no solution, one solution, or infinitely many solutions. There may be more than one possibility for some systems. Justify your answers.

	Number of Equations	Number of Unknowns	Rank of A	Rank of $\left[A \mid \vec{b}\right]$
a.	3	4	—	2
b.	4	3	3	—
c.	4	3	—	4
d.	3	4	3	—

50. Consider a linear system $A\vec{x} = \vec{b}$, where A is a 4×3 matrix. We are told that rank $\left[A \mid \vec{b}\right] = 4$. How many solutions does this system have?

51. Consider an $n \times m$ matrix A, an $r \times s$ matrix B, and a vector $\vec{x}$ in $\mathbb{R}^p$. For which values of n, m, r, s, and p is the product

$$A(B\vec{x})$$

defined?

52. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Can you find a 2×2 matrix C such that

$$A(B\vec{x}) = C\vec{x},$$

for all vectors $\vec{x}$ in $\mathbb{R}^2$?

53. If A and B are two $n \times m$ matrices, is

$$(A + B)\vec{x} = A\vec{x} + B\vec{x}$$

for all $\vec{x}$ in $\mathbb{R}^m$?

54. Consider two vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^3$ that are not parallel. Which vectors in $\mathbb{R}^3$ are linear combinations of $\vec{v}_1$ and $\vec{v}_2$? Describe the set of these vectors geometrically. Include a sketch in your answer.

55. Is the vector $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ a linear combination of

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}?$$

56. Is the vector

$$\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix}$$

a linear combination of

$$\begin{bmatrix} 1 \\ 7 \\ 1 \\ 9 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ -5 \\ 4 \\ 7 \\ 9 \end{bmatrix}?$$

25. Let A be a 4×4 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ is inconsistent. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

26. Let A be a 4×3 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

27. If the rank of a 4×4 matrix A is 4, what is rref(A)?

28. If the rank of a 5×3 matrix A is 3, what is rref(A)?

In Problems 29 through 32, let $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}$ ***and*** $\vec{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

29. Find a *diagonal* matrix A such that $A\vec{x} = \vec{y}$.

30. Find a matrix A of rank 1 such that $A\vec{x} = \vec{y}$.

31. Find an *upper triangular* matrix A such that $A\vec{x} = \vec{y}$. Also, it is required that all the entries of A on and above the diagonal be nonzero.

32. Find a matrix A with all nonzero entries such that $A\vec{x} = \vec{y}$.

33. Let A be the $n \times n$ matrix with all 1's on the diagonal and all 0's above and below the diagonal. What is $A\vec{x}$, where $\vec{x}$ is a vector in $\mathbb{R}^n$?

34. We define the vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^3$.

a. For

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix},$$

compute $A\vec{e}_1$, $A\vec{e}_2$, and $A\vec{e}_3$.

b. If B is an $n \times 3$ matrix with columns $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, what is $B\vec{e}_1$, $B\vec{e}_2$, $B\vec{e}_3$?

35. In $\mathbb{R}^m$, we define

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th component.}$$

If A is an $n \times m$ matrix, what is $A\vec{e}_i$?

36. Find a 3×3 matrix A such that

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},$$

$$\text{and} \quad A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

37. Find all vectors $\vec{x}$ such that $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

38. **a.** Using technology, generate a random 3×3 matrix A. (The entries may be either single-digit integers or numbers between 0 and 1, depending on the technology you are using.) Find rref(A). Repeat this experiment a few times.

b. What does the reduced row-echelon form of most 3×3 matrices look like? Explain.

39. Repeat Exercise 38 for 3×4 matrices.

40. Repeat Exercise 38 for 4×3 matrices.

41. How many solutions do most systems of three linear equations with three unknowns have? Explain in terms of your work in Exercise 38.

42. How many solutions do most systems of three linear equations with four unknowns have? Explain in terms of your work in Exercise 39.

43. How many solutions do most systems of four linear equations with three unknowns have? Explain in terms of your work in Exercise 40.

44. Consider an $n \times m$ matrix A with more rows than columns ($n > m$). Show that there is a vector $\vec{b}$ in $\mathbb{R}^n$ such that the system $A\vec{x} = \vec{b}$ is inconsistent.

45. Consider an $n \times m$ matrix A, a vector $\vec{x}$ in $\mathbb{R}^m$, and a scalar k. Show that

$$A(k\vec{x}) = k(A\vec{x}).$$

46. Find the rank of the matrix

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix},$$

where a, d, and f are nonzero, and b, c, and e are arbitrary numbers.

47. A linear system of the form

$$A\vec{x} = \vec{0}$$

is called *homogeneous*. Justify the following facts:

a. All homogeneous systems are consistent.

5. a. Write the system

$$\begin{vmatrix} x + 2y = 7 \\ 3x + y = 11 \end{vmatrix}$$

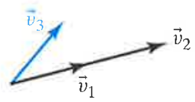
in vector form.

b. Use your answer in part (a) to represent the system geometrically. Solve the system and represent the solution geometrically.

6. Consider the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ in $\mathbb{R}^2$ (sketched in the accompanying figure). Vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel. How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

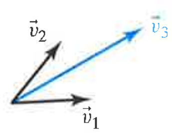
have? Argue geometrically.



7. Consider the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ in $\mathbb{R}^2$ shown in the accompanying sketch. How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

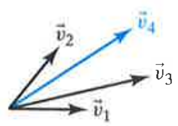
have? Argue geometrically.



8. Consider the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ in $\mathbb{R}^2$ shown in the accompanying sketch. Arguing geometrically, find two solutions x, y, z of the linear system

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{v}_4.$$

How do you know that this system has in fact infinitely many solutions?



9. Write the system

$$\begin{vmatrix} x + 2y + 3z = 1 \\ 4x + 5y + 6z = 4 \\ 7x + 8y + 9z = 9 \end{vmatrix}$$

in matrix form.

Compute the dot products in Exercises 10 through 12 (if the products are defined).

10. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 9 & 9 & 7 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

Compute the products $A\vec{x}$ in Exercises 13 through 15 using paper and pencil. In each case, compute the product two ways: in terms of the columns of A (Theorem 1.3.8) and in terms of the rows of A (Definition 1.3.7).

13. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

Compute the products $A\vec{x}$ in Exercises 16 through 19 using paper and pencil (if the products are defined).

16. $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

20. a. Find $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ 3 & 1 \\ 0 & -1 \end{bmatrix}$.

b. Find $9 \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$.

21. Use technology to compute the product

$$\begin{bmatrix} 1 & 7 & 8 & 9 \\ 1 & 2 & 9 & 1 \\ 1 & 5 & 1 & 5 \\ 1 & 6 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \\ 5 \\ 6 \end{bmatrix}.$$

22. Consider a linear system of three equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

23. Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

24. Let A be a 4×4 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

We can generalize:

Theorem 1.3.11 Matrix form of a linear system

We can write the linear system with augmented matrix $\left[A \;\vdots\; \vec{b}\right]$ in matrix form as

$$A\vec{x} = \vec{b}.$$

Note that the ith component of $A\vec{x}$ is $a_{i1}x_1 + \cdots + a_{im}x_m$, by Definition 1.3.7. Thus, the ith component of the equation $A\vec{x} = \vec{b}$ is

$$a_{i1}x_1 + \cdots + a_{im}x_m = b_i;$$

this is the ith equation of the system with augmented matrix $\left[A \;\vdots\; \vec{b}\right]$.

EXAMPLE 14 Write the system

$$\left| \begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 7 \\ 9x_1 + 4x_2 - 6x_3 &= 8 \end{aligned} \right|$$

in matrix form.

Solution

The coefficient matrix is $A = \begin{bmatrix} 2 & -3 & 5 \\ 9 & 4 & -6 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$. The matrix form is

$$A\vec{x} = \vec{b}, \quad \text{or,} \quad \begin{bmatrix} 2 & -3 & 5 \\ 9 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

Now that we can write a linear system as a *single equation*, $A\vec{x} = \vec{b}$, rather than a list of simultaneous equations, we can think about it in new ways.

For example, if we have an equation $ax = b$ of *numbers*, we can divide both sides by a to find the solution x:

$$x = \frac{b}{a} = a^{-1}b \quad (\text{if } a \neq 0).$$

It is natural to ask whether we can take an analogous approach in the case of the equation $A\vec{x} = \vec{b}$. Can we "divide by A," in some sense, and write

$$\vec{x} = \frac{\vec{b}}{A} = A^{-1}\vec{b}?$$

This issue of the invertibility of a matrix will be one of the main themes of Chapter 2.

EXERCISES 1.3

GOAL *Use the reduced row-echelon form of the augmented matrix to find the number of solutions of a linear system. Apply the definition of the rank of a matrix. Compute the product $A\vec{x}$ in terms of the rows or the columns of A. Represent a linear system in vector or in matrix form.*

1. The reduced row-echelon forms of the augmented matrices of three systems are given below. How many solutions does each system have?

   a. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ **b.** $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 6 \end{array}\right]$

   c. $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Find the rank of the matrices in Exercises 2 through 4.

2. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$