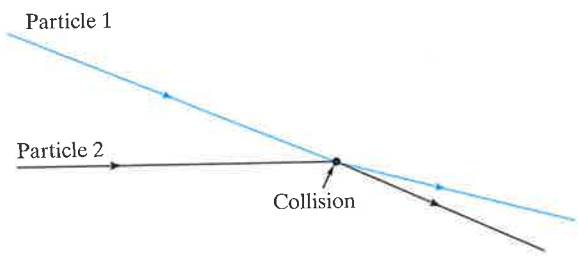
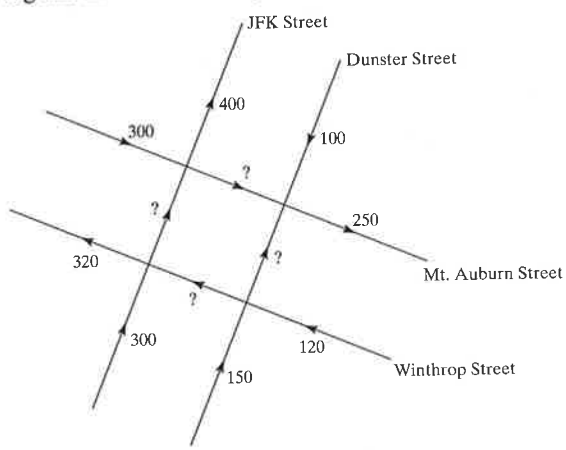
The particles collide. After the collision, their respective velocities are observed to be

$$\vec{w}_1 = \begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{w}_2 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}.$$

Assume that the momentum of the system is conserved throughout the collision. What does this experiment tell you about the masses of the two particles? (See the accompanying figure.)



42. The accompanying sketch represents a maze of one-way streets in a city in the United States. The traffic volume through certain blocks during an hour has been measured. Suppose that the vehicles leaving the area during this hour were exactly the same as those entering it.



What can you say about the traffic volume at the four locations indicated by a question mark? Can you figure out exactly how much traffic there was on each block? If not, describe one possible scenario. For each of the four locations, find the highest and the lowest possible traffic volume.

43. Let $S(t)$ be the length of the tth day of the year 2009 in Mumbai (formerly known as Bombay), India (measured in hours, from sunrise to sunset). We are given the following values of $S(t)$:

t	$S(t)$
47	11.5
74	12
273	12

For example, $S(47) = 11.5$ means that the time from sunrise to sunset on February 16 is 11 hours and 30 minutes. For locations close to the equator, the function $S(t)$ is well approximated by a trigonometric function of the form

$$S(t) = a + b\cos\left(\frac{2\pi t}{365}\right) + c\sin\left(\frac{2\pi t}{365}\right).$$

(The period is 365 days, or 1 year.) Find this approximation for Mumbai, and graph your solution. According to this model, how long is the longest day of the year in Mumbai?

44. Kyle is getting some flowers for Olivia, his Valentine. Being of a precise analytical mind, he plans to spend exactly \$24 on a bunch of exactly two dozen flowers. At the flower market they have lilies (\$3 each), roses (\$2 each), and daisies (\$0.50 each). Kyle knows that Olivia loves lilies; what is he to do?

45. Consider the equations

$$\begin{vmatrix} x + 2y + 3z = 4 \\ x + ky + 4z = 6 \\ x + 2y + (k+2)z = 6 \end{vmatrix},$$

where k is an arbitrary constant.

a. For which values of the constant k does this system have a unique solution?

b. When is there no solution?

c. When are there infinitely many solutions?

46. Consider the equations

$$\begin{vmatrix} y + 2kz = 0 \\ x + 2y + 6z = 2 \\ kx + 2z = 1 \end{vmatrix},$$

where k is an arbitrary constant.

a. For which values of the constant k does this system have a unique solution?

b. When is there no solution?

c. When are there infinitely many solutions?

47. a. Find all solutions x_1, x_2, x_3, x_4 of the system $x_2 = \frac{1}{2}(x_1 + x_3)$, $x_3 = \frac{1}{2}(x_2 + x_4)$.

b. In part (a), is there a solution with $x_1 = 1$ and $x_4 = 13$?

48. For an arbitrary positive integer $n \geq 3$, find all solutions $x_1, x_2, x_3, \ldots, x_n$ of the simultaneous equations $x_2 = \frac{1}{2}(x_1+x_3), x_3 = \frac{1}{2}(x_2+x_4), \ldots, x_{n-1} = \frac{1}{2}(x_{n-2}+x_n)$. Note that we are asked to solve the simultaneous equations $x_k = \frac{1}{2}(x_{k-1} + x_{k+1})$, for $k = 2, 3, \ldots, n-1$.

49. Consider the system

$$\begin{vmatrix} 2x + y = C \\ 3y + z = C \\ x + 4z = C \end{vmatrix},$$

where C is a constant. Find the smallest positive integer C such that x, y, and z are all integers.

The *consumer demand vector* is

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where b_i is the consumer demand on industry I_i. The demand vector for industry I_j is

$$\vec{v}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix},$$

where a_{ij} is the demand industry I_j puts on industry I_i, for each \$1 of output industry I_j produces. For example, $a_{32} = 0.5$ means that industry I_2 needs 50¢worth of products from industry I_3 for each \$1 worth of goods I_2 produces. The coefficient a_{ii} need not be 0: Producing a product may require goods or services from the same industry.

a. Find the four demand vectors for the economy in Exercise 37.

b. What is the meaning in economic terms of $x_j\vec{v}_j$?

c. What is the meaning in economic terms of $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b}$?

d. What is the meaning in economic terms of the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b} = \vec{x}?$$

39. Consider the economy of Israel in 1958.[11] The three industries considered here are

I_1 : agriculture,
I_2 : manufacturing,
I_3 : energy.

Outputs and demands are measured in millions of Israeli pounds, the currency of Israel at that time. We are told that

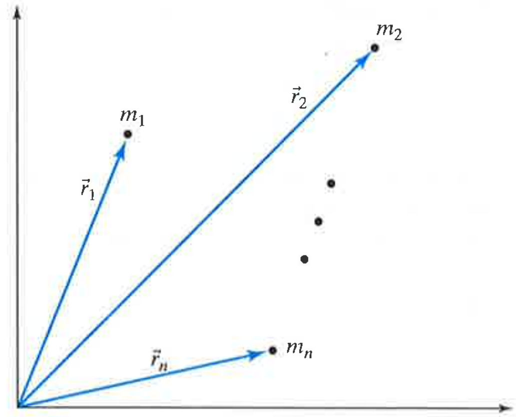
$$\vec{b} = \begin{bmatrix} 13.2 \\ 17.6 \\ 1.8 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 0.293 \\ 0.014 \\ 0.044 \end{bmatrix},$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 0.207 \\ 0.01 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0.017 \\ 0.216 \end{bmatrix}.$$

a. Why do the first components of $\vec{v}_2$ and $\vec{v}_3$ equal 0?

b. Find the outputs x_1, x_2, x_3 required to satisfy demand.

40. Consider some particles in the plane with position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ and masses $m_1, m_2, \ldots, m_n$.

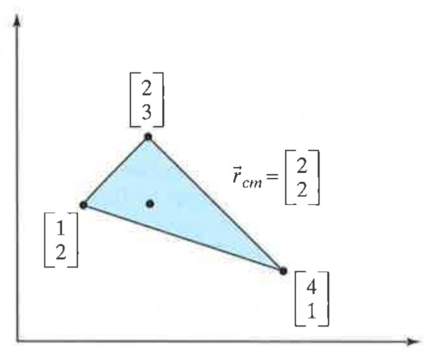


The position vector of the *center of mass* of this system is

$$\vec{r}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n),$$

where $M = m_1 + m_2 + \cdots + m_n$.

Consider the triangular plate shown in the accompanying sketch. How must a total mass of 1 kg be distributed among the three vertices of the plate so that the plate can be supported at the point $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$; that is, $\vec{r}_{cm} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? Assume that the mass of the plate itself is negligible.



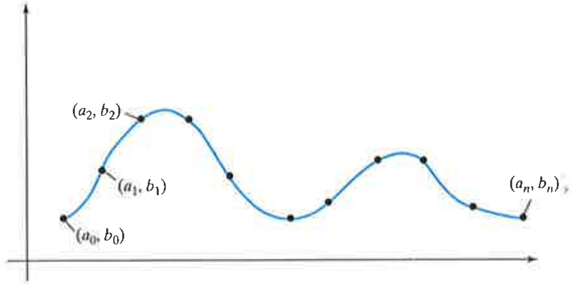
41. The *momentum* $\vec{P}$ of a system of n particles in space with masses $m_1, m_2, \ldots, m_n$ and velocities $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is defined as

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n.$$

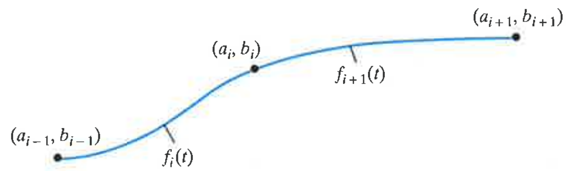
Now consider two elementary particles with velocities

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}.$$

[11] W. Leontief, *Input–Output Economics*, Oxford University Press, 1966.



One method often employed in such design problems is the technique of cubic splines. We choose $f_i(t)$, a polynomial of degree ≤ 3, to define the shape of the ride between (a_{i-1}, b_{i-1}) and (a_i, b_i), for $i = 1, \ldots, n$.



Obviously, it is required that $f_i(a_i) = b_i$ and $f_i(a_{i-1}) = b_{i-1}$, for $i = 1, \ldots, n$. To guarantee a smooth ride at the points (a_i, b_i), we want the first and the second derivatives of f_i and f_{i+1} to agree at these points:

$$\begin{aligned} f_i'(a_i) &= f_{i+1}'(a_i) && \text{and} \\ f_i''(a_i) &= f_{i+1}''(a_i), && \text{for } i = 1, \ldots, n-1. \end{aligned}$$

Explain the practical significance of these conditions. Explain why, for the convenience of the riders, it is also required that

$$f_1'(a_0) = f_n'(a_n) = 0.$$

Show that satisfying all these conditions amounts to solving a system of linear equations. How many variables are in this system? How many equations? (*Note*: It can be shown that this system has a unique solution.)

33. Find the polynomial $f(t)$ of degree 3 such that $f(1) = 1$, $f(2) = 5$, $f'(1) = 2$, and $f'(2) = 9$, where $f'(t)$ is the derivative of $f(t)$. Graph this polynomial.

34. The *dot product* of two vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

in $\mathbb{R}^n$ is defined by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Note that the dot product of two vectors is a scalar. We say that the vectors $\vec{x}$ and $\vec{y}$ are *perpendicular* if $\vec{x} \cdot \vec{y} = 0$.

Find all vectors in $\mathbb{R}^3$ perpendicular to

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

Draw a sketch.

35. Find all vectors in $\mathbb{R}^4$ that are perpendicular to the three vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 9 \\ 9 \\ 7 \end{bmatrix}.$$

(See Exercise 34.)

36. Find all solutions x_1, x_2, x_3 of the equation

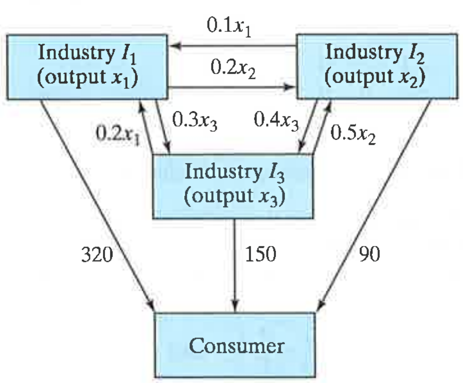
$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3,$$

where

$$\vec{b} = \begin{bmatrix} -8 \\ -1 \\ 2 \\ 15 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ 6 \\ 9 \\ 1 \end{bmatrix}.$$

37. For some background on this exercise, see Exercise 1.1.20.

Consider an economy with three industries, I_1, I_2, I_3. What outputs x_1, x_2, x_3 should they produce to satisfy both consumer demand and interindustry demand? The demands put on the three industries are shown in the accompanying figure.



38. If we consider more than three industries in an input–output model, it is cumbersome to represent all the demands in a diagram as in Exercise 37. Suppose we have the industries $I_1, I_2, \ldots, I_n$, with outputs $x_1, x_2, \ldots, x_n$. The *output vector* is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

12. $\left|\begin{array}{l} 2x_1 \quad - 3x_3 \quad + 7x_5 + 7x_6 = 0 \\ -2x_1 + x_2 + 6x_3 \quad - 6x_5 - 12x_6 = 0 \\ x_2 - 3x_3 \quad + x_5 + 5x_6 = 0 \\ - 2x_2 \quad + x_4 + x_5 + x_6 = 0 \\ 2x_1 + x_2 - 3x_3 \quad + 8x_5 + 7x_6 = 0 \end{array}\right|$

Solve the linear systems in Exercises 13 through 17. You may use technology.

13. $\left|\begin{array}{l} 3x + 11y + 19z = -2 \\ 7x + 23y + 39z = 10 \\ -4x - 3y - 2z = 6 \end{array}\right|$

14. $\left|\begin{array}{l} 3x + 6y + 14z = 22 \\ 7x + 14y + 30z = 46 \\ 4x + 8y + 7z = 6 \end{array}\right|$

15. $\left|\begin{array}{l} 3x + 5y + 3z = 25 \\ 7x + 9y + 19z = 65 \\ -4x + 5y + 11z = 5 \end{array}\right|$

16. $\left|\begin{array}{l} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 = 11 \end{array}\right|$

17. $\left|\begin{array}{l} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 = 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 = 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 = 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 = 24 \end{array}\right|$

18. Determine which of the matrices below are in reduced row-echelon form:

a. $\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ **d.** $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}$

19. Find all 4×1 matrices in reduced row-echelon form.

20. We say that two $n \times m$ matrices in reduced row-echelon form are of the same type if they contain the same number of leading 1's in the same positions. For example,

$$\begin{bmatrix} \textcircled{1} & 2 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \textcircled{1} & 3 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

are of the same type. How many types of 2×2 matrices in reduced row-echelon form are there?

21. How many types of 3×2 matrices in reduced row-echelon form are there? (See Exercise 20.)

22. How many types of 2×3 matrices in reduced row-echelon form are there? (See Exercise 20.)

23. Suppose you apply Gauss–Jordan elimination to a matrix. Explain how you can be sure that the resulting matrix is in reduced row-echelon form.

24. Suppose matrix A is transformed into matrix B by means of an elementary row operation. Is there an elementary row operation that transforms B into A? Explain.

25. Suppose matrix A is transformed into matrix B by a sequence of elementary row operations. Is there a sequence of elementary row operations that transforms B into A? Explain your answer. (See Exercise 24.)

26. Consider an $n \times m$ matrix A. Can you transform $\operatorname{rref}(A)$ into A by a sequence of elementary row operations? (See Exercise 25.)

27. Is there a sequence of elementary row operations that transforms

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ?$$

Explain.

28. Suppose you subtract a multiple of an equation in a system from another equation in the system. Explain why the two systems (before and after this operation) have the same solutions.

29. *Balancing a chemical reaction.* Consider the chemical reaction

$$a\ NO_2 + b\ H_2O \rightarrow c\ HNO_2 + d\ HNO_3,$$

where a, b, c, and d are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must be the same before and after the reaction. For example, because the number of oxygen atoms must remain the same,

$$2a + b = 2c + 3d.$$

While there are many possible values for a, b, c, and d that balance the reaction, it is customary to use the smallest possible positive integers. Balance this reaction.

30. Find the polynomial of degree 3 [a polynomial of the form $f(t) = a + bt + ct^2 + dt^3$] whose graph goes through the points $(0, 1)$, $(1, 0)$, $(-1, 0)$, and $(2, -15)$. Sketch the graph of this cubic.

31. Find the polynomial of degree 4 whose graph goes through the points $(1, 1)$, $(2, -1)$, $(3, -59)$, $(-1, 5)$, and $(-2, -29)$. Graph this polynomial.

32. *Cubic splines.* Suppose you are in charge of the design of a roller coaster ride. This simple ride will not make any left or right turns; that is, the track lies in a vertical plane. The accompanying figure shows the ride as viewed from the side. The points (a_i, b_i) are given to you, and your job is to connect the dots in a reasonably smooth way. Let $a_{i+1} > a_i$.

squares, which he had developed around 1794. (See Section 5.4.) Since Gauss at first refused to reveal the methods that led to this amazing accomplishment, some even accused him of sorcery. Gauss later described his methods of orbit computation in his book *Theoria Motus Corporum Coelestium* (1809).

The method of solving a linear system by Gauss–Jordan elimination is called an *algorithm*.[10] An algorithm can be defined as "a finite procedure, written in a fixed symbolic vocabulary, governed by precise instructions, moving in discrete Steps, 1, 2, 3, . . . , whose execution requires no insight, cleverness, intuition, intelligence, or perspicuity, and that sooner or later comes to an end" (David Berlinski, *The Advent of the Algorithm: The Idea that Rules the World*, Harcourt Inc., 2000).

Gauss–Jordan elimination is well suited for solving linear systems on a computer, at least in principle. In practice, however, some tricky problems associated with roundoff errors can occur.

Numerical analysts tell us that we can reduce the proliferation of roundoff errors by modifying Gauss–Jordan elimination, employing more sophisticated reduction techniques.

In modifying Gauss–Jordan elimination, an interesting question arises: If we transform a matrix A into a matrix B by a sequence of elementary row operations and if B is in reduced row-echelon form, is it necessarily true that $B = \operatorname{rref}(A)$? Fortunately (and perhaps surprisingly) this is indeed the case.

In this text, we will not utilize this fact, so there is no need to present the somewhat technical proof. If you feel ambitious, try to work out the proof yourself after studying Chapter 3. (See Exercises 3.3.84 through 3.3.87.)

[10] The word *algorithm* is derived from the name of the mathematician al-Khowarizmi, who introduced the term *algebra* into mathematics. (See page 1.)

EXERCISES 1.2

GOAL *Use Gauss–Jordan elimination to solve linear systems. Do simple problems using paper and pencil, and use technology to solve more complicated problems.*

In Exercises 1 through 12, find all solutions of the equations with paper and pencil using Gauss–Jordan elimination. Show all your work. Solve the system in Exercise 8 for the variables x_1, x_2, x_3, x_4, and x_5.

1. $\left| \begin{array}{l} x + y - 2z = 5 \\ 2x + 3y + 4z = 2 \end{array} \right|$

2. $\left| \begin{array}{l} 3x + 4y - z = 8 \\ 6x + 8y - 2z = 3 \end{array} \right|$

3. $x + 2y + 3z = 4$

4. $\left| \begin{array}{l} x + y = 1 \\ 2x - y = 5 \\ 3x + 4y = 2 \end{array} \right|$

5. $\left| \begin{array}{l} x_3 + x_4 = 0 \\ x_2 + x_3 = 0 \\ x_1 + x_2 = 0 \\ x_1 + x_4 = 0 \end{array} \right|$

6. $\left| \begin{array}{l} x_1 - 7x_2 + x_5 = 3 \\ x_3 - 2x_5 = 2 \\ x_4 + x_5 = 1 \end{array} \right|$

7. $\left| \begin{array}{l} x_1 + 2x_2 \quad 2x_4 + 3x_5 = 0 \\ x_3 + 3x_4 + 2x_5 = 0 \\ x_3 + 4x_4 - x_5 = 0 \\ x_5 = 0 \end{array} \right|$

8. $\left| \begin{array}{l} x_2 + 2x_4 + 3x_5 = 0 \\ 4x_4 + 8x_5 = 0 \end{array} \right|$

9. $\left| \begin{array}{l} x_4 + 2x_5 - x_6 = 2 \\ x_1 + 2x_2 + x_5 - x_6 = 0 \\ x_1 + 2x_2 + 2x_3 - x_5 + x_6 = 2 \end{array} \right|$

10. $\left| \begin{array}{l} 4x_1 + 3x_2 + 2x_3 - x_4 = 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 = 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 = -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 = 11 \end{array} \right|$

11. $\left| \begin{array}{l} x_1 + 2x_3 + 4x_4 = -8 \\ x_2 - 3x_3 - x_4 = 6 \\ 3x_1 + 4x_2 - 6x_3 + 8x_4 = 0 \\ -x_2 + 3x_3 + 4x_4 = -12 \end{array} \right|$