the points $(1, p)$, $(2, q)$, $(3, r)$, where p, q, r are arbitrary constants. Does such a polynomial exist for all values of p, q, r?

31. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 3)$ and $(2, 6)$, such that $f'(1) = 1$ [where $f'(t)$ denotes the derivative].

32. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 1)$ and $(2, 0)$, such that $\int_1^2 f(t)\,dt = -1$.

33. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 1)$ and $(3, 3)$, such that $f'(2) = 1$.

34. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 1)$ and $(3, 3)$, such that $f'(2) = 3$.

35. Find the function $f(t)$ of the form $f(t) = ae^{3t} + be^{2t}$ such that $f(0) = 1$ and $f'(0) = 4$.

36. Find the function $f(t)$ of the form $f(t) = a\cos(2t) + b\sin(2t)$ such that $f''(t) + 2f'(t) + 3f(t) = 17\cos(2t)$. (This is the kind of differential equation you might have to solve when dealing with forced damped oscillators, in physics or engineering.)

37. Find the circle that runs through the points $(5, 5)$, $(4, 6)$, and $(6, 2)$. Write your equation in the form $a + bx + cy + x^2 + y^2 = 0$. Find the center and radius of this circle.

38. Find the ellipse centered at the origin that runs through the points $(1, 2)$, $(2, 2)$, and $(3, 1)$. Write your equation in the form $ax^2 + bxy + cy^2 = 1$.

39. Find all points (a, b, c) in space for which the system

$$\left| \begin{array}{l} x + 2y + 3z = a \\ 4x + 5y + 6z = b \\ 7x + 8y + 9z = c \end{array} \right|$$

has at least one solution.

40. Linear systems are particularly easy to solve when they are in triangular form (i.e., all entries above or below the diagonal are zero).

a. Solve the lower triangular system

$$\left| \begin{array}{rcl} x_1 & = & -3 \\ -3x_1 + x_2 & = & 14 \\ x_1 + 2x_2 + x_3 & = & 9 \\ -x_1 + 8x_2 - 5x_3 + x_4 & = & 33 \end{array} \right|$$

by forward substitution, finding x_1 first, then x_2, then x_3, and finally x_4.

b. Solve the upper triangular system

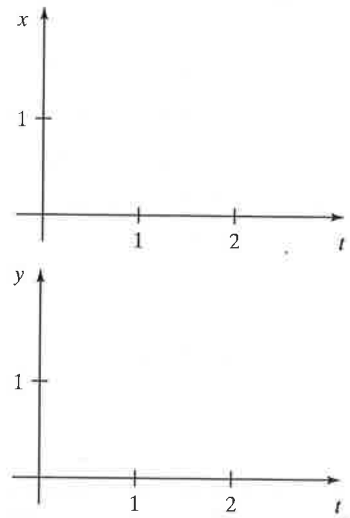
$$\left| \begin{array}{rcl} x_1 + 2x_2 - x_3 + 4x_4 & = & -3 \\ x_2 + 3x_3 + 7x_4 & = & 5 \\ x_3 + 2x_4 & = & 2 \\ x_4 & = & 0 \end{array} \right|.$$

41. Consider the linear system

$$\left| \begin{array}{l} x + y = 1 \\ x + \frac{t}{2}y = t \end{array} \right|,$$

where t is a nonzero constant.

a. Determine the x- and y-intercepts of the lines $x + y = 1$ and $x + (t/2)y = t$; sketch these lines. For which values of the constant t do these lines intersect? For these values of t, the point of intersection (x, y) depends on the choice of the constant t; that is, we can consider x and y as functions of t. Draw rough sketches of these functions.



Explain briefly how you found these graphs. Argue geometrically, without solving the system algebraically.

b. Now solve the system algebraically. Verify that the graphs you sketched in part (a) are compatible with your algebraic solution.

42. Find a system of linear equations with three unknowns whose solutions are the points on the line through $(1, 1, 1)$ and $(3, 5, 0)$.

43. Find a system of linear equations with three unknowns x, y, z whose solutions are

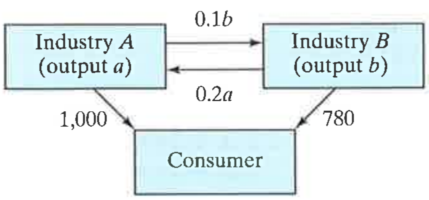
$$x = 6 + 5t, \quad y = 4 + 3t, \quad \text{and} \quad z = 2 + t,$$

where t is an arbitrary constant.

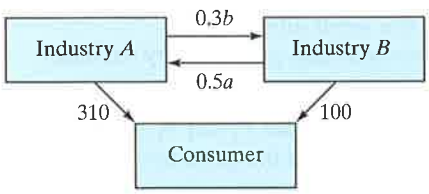
44. Boris and Marina are shopping for chocolate bars. Boris observes, "If I add half my money to yours, it will be enough to buy two chocolate bars." Marina naively asks, "If I add half my money to yours, how many can we buy?" Boris replies, "One chocolate bar." How much money did Boris have? (From Yuri Chernyak and Robert Rose, *The Chicken from Minsk*, Basic Books, 1995.)

45. Here is another method to solve a system of linear equations: Solve one of the equations for one of the variables, and substitute the result into the other equations. Repeat

You may be tempted to say 1,000 and 780, respectively, but things are not quite as simple as that. We have to take into account the interindustry demand as well. Let us say that industry A produces electricity. Of course, producing almost any product will require electric power. Suppose that industry B needs 10¢worth of electricity for each $1 of output B produces and that industry A needs 20¢worth of B's products for each $1 of output A produces. Find the outputs a and b needed to satisfy both consumer and interindustry demand.



21. Find the outputs a and b needed to satisfy the consumer and interindustry demands given in the following figure (see Exercise 20):



22. Consider the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - x = \cos(t).$$

This equation could describe a forced damped oscillator, as we will see in Chapter 9. We are told that the differential equation has a solution of the form

$$x(t) = a\sin(t) + b\cos(t).$$

Find a and b, and graph the solution.

23. Find all solutions of the system

$$\begin{vmatrix} 7x - y = \lambda x \\ -6x + 8y = \lambda y \end{vmatrix}, \quad \text{for}$$

a. $\lambda = 5$ **b.** $\lambda = 10$, and **c.** $\lambda = 15$.

24. On your next trip to Switzerland, you should take the scenic boat ride from Rheinfall to Rheinau and back. The trip downstream from Rheinfall to Rheinau takes 20 minutes, and the return trip takes 40 minutes; the distance between Rheinfall and Rheinau along the river is 8 kilometers. How fast does the boat travel (relative to the water), and how fast does the river Rhein flow in this area? You may assume both speeds to be constant throughout the journey.

25. Consider the linear system

$$\begin{vmatrix} x + y - z = -2 \\ 3x - 5y + 13z = 18 \\ x - 2y + 5z = k \end{vmatrix},$$

where k is an arbitrary number.

a. For which value(s) of k does this system have one or infinitely many solutions?

b. For each value of k you found in part a, how many solutions does the system have?

c. Find all solutions for each value of k.

26. Consider the linear system

$$\begin{vmatrix} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + (k^2 - 5)z = k \end{vmatrix},$$

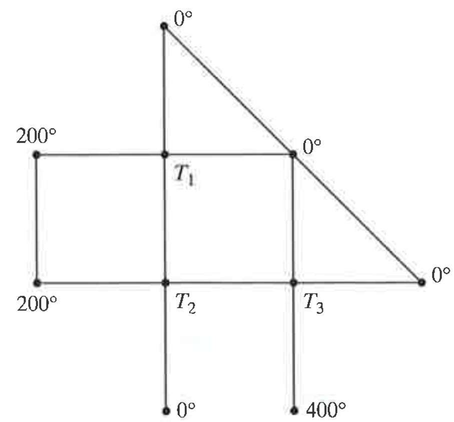
where k is an arbitrary constant. For which value(s) of k does this system have a unique solution? For which value(s) of k does the system have infinitely many solutions? For which value(s) of k is the system inconsistent?

27. Emile and Gertrude are brother and sister. Emile has twice as many sisters as brothers, and Gertrude has just as many brothers as sisters. How many children are there in this family?

28. In a grid of wires, the temperature at exterior mesh points is maintained at constant values (in °C) as shown in the accompanying figure. When the grid is in thermal equilibrium, the temperature T at each interior mesh point is the average of the temperatures at the four adjacent points. For example,

$$T_2 = \frac{T_3 + T_1 + 200 + 0}{4}.$$

Find the temperatures T_1, T_2, and T_3 when the grid is in thermal equilibrium.



29. Find the polynomial of degree 2 [a polynomial of the form $f(t) = a + bt + ct^2$] whose graph goes through the points $(1, -1)$, $(2, 3)$, and $(3, 13)$. Sketch the graph of this polynomial.

30. Find a polynomial of degree ≤ 2 [a polynomial of the form $f(t) = a + bt + ct^2$] whose graph goes through

A System without Solutions

In the following system, perform the eliminations yourself to obtain the result shown:

$$\left| \begin{array}{l} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 0 \end{array} \right| \longrightarrow \left| \begin{array}{l} x \quad - z = 2 \\ y + 2z = -1 \\ 0 = -6 \end{array} \right|.$$

Whatever values we choose for x, y, and z, the equation $0 = -6$ cannot be satisfied. This system is *inconsistent*; that is, it has no solutions.

EXERCISES 1.1

GOAL *Set up and solve systems with as many as three linear equations with three unknowns, and interpret the equations and their solutions geometrically.*

In Exercises 1 through 10, find all solutions of the linear systems using elimination as discussed in this section. Then check your solutions.

1. $\left| \begin{array}{l} x + 2y = 1 \\ 2x + 3y = 1 \end{array} \right|$

2. $\left| \begin{array}{l} 4x + 3y = 2 \\ 7x + 5y = 3 \end{array} \right|$

3. $\left| \begin{array}{l} 2x + 4y = 3 \\ 3x + 6y = 2 \end{array} \right|$

4. $\left| \begin{array}{l} 2x + 4y = 2 \\ 3x + 6y = 3 \end{array} \right|$

5. $\left| \begin{array}{l} 2x + 3y = 0 \\ 4x + 5y = 0 \end{array} \right|$

6. $\left| \begin{array}{l} x + 2y + 3z = 8 \\ x + 3y + 3z = 10 \\ x + 2y + 4z = 9 \end{array} \right|$

7. $\left| \begin{array}{l} x + 2y + 3z = 1 \\ x + 3y + 4z = 3 \\ x + 4y + 5z = 4 \end{array} \right|$

8. $\left| \begin{array}{l} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 10z = 0 \end{array} \right|$

9. $\left| \begin{array}{l} x + 2y + 3z = 1 \\ 3x + 2y + z = 1 \\ 7x + 2y - 3z = 1 \end{array} \right|$

10. $\left| \begin{array}{l} x + 2y + 3z = 1 \\ 2x + 4y + 7z = 2 \\ 3x + 7y + 11z = 8 \end{array} \right|$

In Exercises 11 through 13, find all solutions of the linear systems. Represent your solutions graphically, as intersections of lines in the x–y-plane.

11. $\left| \begin{array}{l} x - 2y = 2 \\ 3x + 5y = 17 \end{array} \right|$

12. $\left| \begin{array}{l} x - 2y = 3 \\ 2x - 4y = 6 \end{array} \right|$

13. $\left| \begin{array}{l} x - 2y = 3 \\ 2x - 4y = 8 \end{array} \right|$

In Exercises 14 through 16, find all solutions of the linear systems. Describe your solutions in terms of intersecting planes. You need not sketch these planes.

14. $\left| \begin{array}{l} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 1 \end{array} \right|$

15. $\left| \begin{array}{l} x + y - z = 0 \\ 4x - y + 5z = 0 \\ 6x + y + 4z = 0 \end{array} \right|$

16. $\left| \begin{array}{l} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 0 \end{array} \right|$

17. Find all solutions of the linear system

$$\left| \begin{array}{l} x + 2y = a \\ 3x + 5y = b \end{array} \right|,$$

where a and b are arbitrary constants.

18. Find all solutions of the linear system

$$\left| \begin{array}{l} x + 2y + 3z = a \\ x + 3y + 8z = b \\ x + 2y + 2z = c \end{array} \right|,$$

where a, b, and c are arbitrary constants.

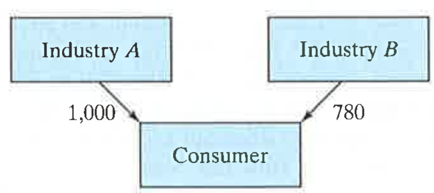
19. Consider a two-commodity market. When the unit prices of the products are P_1 and P_2, the quantities demanded, D_1 and D_2, and the quantities supplied, S_1 and S_2, are given by

$$D_1 = 70 - 2P_1 + P_2, \quad S_1 = -14 + 3P_1,$$
$$D_2 = 105 + P_1 - P_2, \quad S_2 = -7 + 2P_2.$$

a. What is the relationship between the two commodities? Do they compete, as do Volvos and BMWs, or do they complement one another, as do shirts and ties?

b. Find the equilibrium prices (i.e., the prices for which supply equals demand), for both products.

20. The Russian-born U.S. economist and Nobel laureate Wassily Leontief (1906–1999) was interested in the following question: What output should each of the industries in an economy produce to satisfy the total demand for all products? Here, we consider a very simple example of input-output analysis, an economy with only two industries, A and B. Assume that the consumer demand for their products is, respectively, 1,000 and 780, in millions of dollars per year.



What outputs a and b (in millions of dollars per year) should the two industries generate to satisfy the demand?