MATH 3406M (McCuan) Spring 2022
Some Administrative Details

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I have tried to record here some information related to the course *A Second Course in Linear Algebra* given Spring semester of 2022 at Georgia Tech.[1] This information is organized in two sections entitled "Stuff" and "Non-Stuff." Perhaps the most important thing for actually taking the course, if one is going to take the course in order to learn linear algebra, is the course page, which I hope remains available to students. This "course page" is made available, at least at the moment, through the long nightmare commonly known as the internet, and the url (or internet address) is

`http://www.math.gatech.edu/~mccuan/courses/3406`

I'm really excited about teaching linear algebra this semester, and I think there is an opportunity for you to learn many interesting and exciting things. It should be fun.

The basic message of the second section "Non-Stuff" is a kind of request: If you anticipate asking me questions about any of the topics discussed in this section during the semester, then please login to OSCAR (or wherever you register for classes) and remove this course from your schedule. You can perhaps take the same course from a different instructor or if that is not possible, you can take the same course another semester. It is not particularly that I am "offended" by such questions, but rarely if ever am I able to offer the kind of answer a student who asks such questions finds satisfying. On the contrary, my answer to students who ask such questions tends to make such students uncomfortable and sometimes leads them to pursue behaviors which can potentially make me uncomfortable. So why should any of us deal with all this discomfort? Just sign up for a different course now. If you have seen the movie "The Matrix," then you can perhaps understand my request. I'm sort of like the glitch in the matrix. If you want to maintain the illusion of the matrix and keep the wool pulled down tightly over your eyes, then you should avoid people like me.

1 Stuff

1.1 The Textbook

I am planning to use (primarily) the textbook *Linear Algebra Done Right* by Sheldon Axler. I was working nearby San Francisco, where Sheldon Axler works, when he was working on this textbook. He was around from time to time, and you can say I sort of knew him a little bit. At least I saw him around and knew who he was. When the textbook first came out I was rather put off by the title, and as a consequence I have avoided the textbook until now (some 20 or so years later). This is technically the first time I've taught this course, or any 3000 (junior) level math course at Georgia Tech for that matter, though I've taught

[1] It should be noted that this document refers exclusively to the "section" of the course taught by John McCuan. There are other sections of the same course with the same title and subject matter being taught the same semester by a different instructor (or different instructors), and nothing in this document is intended to apply to those sections of the course.

a course MATH 4305 several times, which I think amounts to basically the same thing. When I taught 4305 before I used either Strang or Curtis, both of which are pretty good linear algebra textbooks—and in some sense my favorite is Curtis. In any case, when I looked over the course description for MATH 3406, I noted that it specifically mentioned Axler's book and seemed to have been designed specifically around that text, so I decided to have a look.

There are lots and lots of really lousy linear algebra textbooks, which I've long thought is a little bit strange because linear algebra is a fairly easy subject, and in principle it contains some pretty compelling ideas—it should be pretty easy to motivate and so it *should* be a subject which it is easy for students to get motivated about...and learn. Having said that, I didn't feel the same way about it when I first took a linear algebra course some 30 or so years ago. I thought the subject was just awful. It wasn't until I understood the subject (an embarrassing number of years later) that I came to appreciate it and get some perspective. Of course, the textbook from which my first course was taught was one of the absolutely terrible ones, and reading Curtis was quite instrumental for me in actually understanding the subject, so that may explain why Curtis is my favorite. I'll mention one more thing on this kind of aside: Once I was really excited because a famous mathematician, Peter Lax, whom I knew personally and knew to be an amazing expositor was writing a textbook which promised to present linear algebra motivated from the perspective of computer graphics, which sounds like a wonderful idea as much of 3D computer graphics (and the 2D theory of vision) is basically just linear algebra. Maybe it is a wonderful idea, but when I tried to read Lax' book, I was really disappointed. That taught me that while linear algebra *should* be easy and easy to motivate, practically the situation is somewhat otherwise, especially with the motivation, which I'll talk about a little more below.

Returning to Axler's book, I'm happy to say I was pleasantly surprised. In fact, I am quite hopeful that after this semester I'll be able to say *Linear Algebra Done Right* is another of my favorite linear algebra textbooks. I still recoil from the title, but I would say the book can truly claim to be *Linear Algebra Done Well,* which is pretty good. The problem is that each individual learns in a different way, so there really can be no "right" way to present particular concepts or a particular subject. That is just wrong. I still think that if linear algebra were done "right" if such a thing were possible, then one would include some compelling material motivating the subject—something that gets across to the student, or at least has a chance of getting across to the student—some reason for actually making the investment to learn the material of the course. This might be accomplished through a presentation of applications like computer graphics or some kind of geometry or art or even building the course around such an application as Lax imagined. This is presumably why Strang is still explicitly included in the course description for MATH 3406. It's probably fair to say that Strang is not completely successful with regard to motivation either, but at least he tries giving some applications of eigenvalues and eigenvectors as well as some applications of orthogonality to plane geometry, and what Strang has produced is perhaps the best available in this direction. The fact remains that no one seems to have been really successful is producing reasonable motivational material for linear algebra. Axler has nothing of the sort—no motivation whatsoever. He assumes the student is already "with the program" and is somehow magically motivated to learn linear algebra. I will make some kind of modest attempt at motivation below. I hope you find it motivating, but it assumes you know about functions generally and specifically about calculus. We'll see.

What Axler gets right: Axler has produced an exceptionally well-organized and clear book. This is really the strong point. Linear algebra is a bit complicated, especially at certain points, and Axler presents essentially all of these really nicely. So I think one can get a pretty good "high level" feel for linear algebra, which probably offers pretty good functionality/facility. There is a danger, I think, in such a presentation that things become too "slick," and one doesn't actually apreciate the complications for what they are. More generally, there is a fashion in exposition these days toward slick presentation that makes the student "feel" like he understands the material—without actually understanding it. But if one likes that sort of thing—and most students do—then Axler is really good at it. It is clear he can do this because he has an exceptionally good understanding of linear algebra from the purely mathematical point of view, and specifically from the somewhat more advanced perspective of functional analysis of linear operators, to which he makes explicit reference sometimes. That is all very good. I'm guessing Axler is a bit weak on his perspective on applications, which explains why he does not include much of any or give any real motivation.

Another big thing that Axler gets right (though it might seem very simple and like a "no-brainer") is what the subject is actually about, namely **the study of linear functions on finite dimensional vector spaces** and on finite dimensional Euclidean spaces in particular. Most linear algebra textbooks come across as being about matrices or solving systems of linear equations. These are, of course, topics that are included in linear algebra, but they are not really what linear algebra is about, in my humble opinion—and that of Axler. In any case, Axler and I agree on this. There are many linear algebra textbooks that make linear algebra out to be about matrices and/or systems of linear equations, and that is just wrong; these are the lousy linear algebra textbooks I mentioned above. In some sense, one can say that Strang turns this around and, while understanding what the subject is about, uses matrices and systems of linear equations as side props to present the subject. It's not necessarily ideal, but Strang is successful in pulling it off.

What Axler gets wrong: Axler makes a big deal about **determinants** and not discussing them first yada yada yada. Determinants are another topic that, while not really the main topic of linear algebra, are (1) important, (2) relatively easy to motivate, and (3) fun. So there's really no need for all the hating on determinants. Having said that, my mentor in teaching (who I was working with at the time I was around Axler) once told me: If anyone ever tries to teach determinants and does not start with the words "area" and "volume," that person should immediately be taken out in back of the university and shot. Axler fails miserably on this one, introducing the determinant simply as an "invariant" of a linear transformation, which is true but not very enlightening. This is somewhat mitigated by the fact that Axler covers determinants at the end of the book making them a topic which is sort of relegated to the end of the course which no one is going to think much about. It's not really an excuse, but that's the way it is. I may try to remedy this situation myself, but determinants is another topic which is usually not done well in linear algebra textbooks. If I do it, rest assured I'll start with "area" and "volume."

1.2 The Assignments

I'm planning to type up homework assignments. Some of them will be designated "exams" simply because there is some "regulation" that I'm required to give exams. There is really no difference between exams and other homework assignments. If you want to learn the subject, you need to work problems. You need to think about things. You need to write down an expression of your thinking clearly and see if someone else can understand what you've written. It also helps to talk about what you are thinking—if you can express it in some kind of relatively comprehensible way in speech. You will have some chance to do that in the classroom during the lectures. But the assignments give you a chance to write things down and have someone try to read it.

1.3 The Lectures

I guess I will give lectures. These are not the most useful things. During the lectures (if you are there) I will ask you what you are thinking about, and what you want to discuss. If you have nothing to say, this probably means you are not thinking. If you are not thinking, you are not learning. I cannot learn anything for you. I cannot learn you anything. You must learn yourself, if you want to do so. In order to learn yourself, you must think yourself. I can't do that for you either. Any instructor who makes you "feel" like you've learned something from listening to an entertaining lecture is deceiving you and doing you a disservice. Any instructor who makes you feel like you've learned something because you can jump through a bunch of hoops using mindless techniques and formulas is tricking you too and actually **"training" you to be unable to think**. Organizing courses built around this sort of mindlessness, you may notice, is what passes for "good teaching." It makes students feel good and write nice things on student satisfaction surveys.

The uncomfortable truth is that, for most students, thinking is an unfamiliar and unwelcome activity. I don't know what else to say about that. I think it's important for people to think and to think critically. For me, that's the primary reason to teach courses. It's not really about linear algebra or calculus—it's about being able to think—mathematics is just a tool to offer the opportunity to think. That's why I'm here: To give you the opportunity (other instructors have not) to think. But I can't do it for you. For one who can think (and think critically) the aggregate effect of being "trained" not to think and to embrace every technology and whatever is regulated and mandated is pretty obvious. There are people working in labs to make viruses more contagious and more deadly; there are people in labs developing nuclear bombs and nuclear reactors which can kill tens of thousands of people within seconds and pollute areas on the earth for generations. It is clear that the "training" students receive in mindless hoop-jumping and non-thinking has produced this situation. I don't know if getting some people to think can reverse the direction, but I can't think of a better idea at the moment.

Of course, maybe you think everything is lollipops and skittle colored unicorns and things are just getting better and better and better (through technology and compliance with regulations and people doing what they are told to do), and you are going to go out and make the lives of all people better and better and better...through mindless hoop-jumping and the "knowledge economy"...of the future. Well, we just have a difference of opinion. But perhaps if you're really committed to the mindless hoop-jumping you should find a different instructor.

1.4 Feedback

Now, looking on the bright side, let's assume you are going to think about linear algebra and try to learn something this semester. When you do that, it can be fun and helpful to have some feedback. This is mostly not applauding and handing out gold stars. It's having someone read carefully what you've written and **complaining** about it. I'm going to apologize up front that I cannot offer you more feedback. This is something that you just might be justified in complaining about, and it is certainly something that is worth doing something about. There are a couple reasons for the difficulty in receiving feedback. One is that there are so many of you and so few of me. There are very few voices around here advocating for classes with 10 students instead of 40 (or 250). And in all honesty "training" can work just about as well with 250 as with 10—actual thinking and learning not so much. The other factor is that the typical student has never learned to express himself so clearly, and trying to read unintelligible gibberish is something no one wants to do for so long. I can help with this latter aspect, but it takes time. Once you can write pretty clearly that will help. But you need to work for it. That means (probably) that you'll need to bring your paper to me personally (after you get it back from the grader) and ask me to read it.

It is here that your real job begins: First you have to think. Then you have to have something to say. Then you need to express it clearly. Then you write it down, bring it to me, and I'll try to have a look and complain about it. Part of the job will already be done because when you did the thinking and careful writing, you were providing your own feedback. Ideally, your expectation for me at that point will rather be just about how to express your ideas clearly rather than: "I don't know how to do this problem. I don't know how to start." When you find your own starting point and have your own questions and thoughts, you are starting to give your own feedback and preparing yourself for more serious feedback from someone else. Another really good thing you can and should do in this regard is to speak up in class and even present problems (and your solutions or attempted solutions). As I mentioned above: I will ask you what you're thinking about. This is the time to tell me. It will be fun.

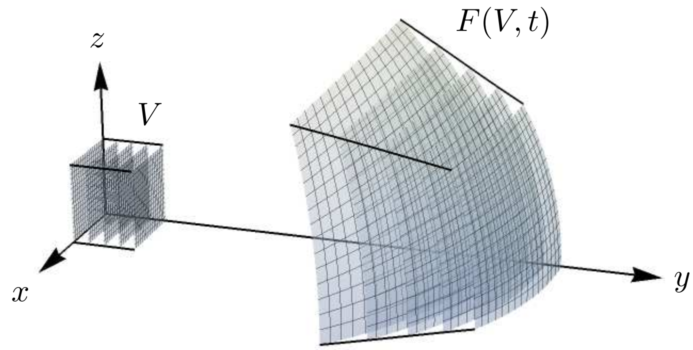

Figure 1: A rectangular box volume V moving and deforming in space under a flow.

1.5 Why Study Linear Albebra?

I am now going to attempt to explain why I eventually decided linear algebra was worth learning and sat down and learned it. What served me for motivation may not be compelling at all for you. If not, I'm sorry about that and probably the best thing for you to do is not worry about it until you figure out some good answer to the question posed by the title of this subsection.[2]

First of all, I like calculus. Part of calculus is about taking derivatives which measure rates of change. I like derivatives and especially **geometric** rates of change. For example, the derivative of a real valued function of one variable gives the **slope** of the tangent line to the graph of that function. Along with this is the basic idea that the behavior of the function near a given point can be approximated by what is called the **linear approximation**. In symbols: If $f : (a, b) \to \mathbb{R}$ is a real valued function of one variable defined on an interval (a, b) and $a < x_0 < b$, then to "zero order" $f(x)$ is approximately $f(x_0)$, the value of f at $x = x_0$, at least as long as f is **continuous** at x_0. More importantly, if f is **differentiable** at $x = x_0$, then the values of $f(x)$ for x near x_0 can be approximated by a formula

$$f(x) \sim f(x_0) + L(x - x_0)$$

where $L : \mathbb{R} \to \mathbb{R}$ is the **linear function** given by $L(v) = f'(x_0)v$. This is called the **first order approximation formula**, and to a certain extent this formula is central to calculus. (And I like calculus.)

Now, what I have said above doesn't really get you to linear algebra. However, there is another kind of calculus problem I thought was really compelling. Say you have a volume V in three-dimensional Euclidean space. Maybe V is a rectangular box as indicated in Figure 1. You can imagine the box V is filled with some gas or other compressible substance and, imagine also, that this volume of substance **flows** in space with maybe changing relative positions and densities. One way to model this is to associated with each position $(x, y, z) \in V$ a subsequent position at a later time $t > 0$. We can call the point to which (x, y, z) arrives after time t as a result of the flow

$$F(x, y, z, t).$$

All the points together that started in V should comprise some kind of (probably deformed) volume $F(V, t)$ at time t. One possible, relatively simple, kind of deformed volume $F(V, t)$ is indicated to the right in Figure 1. But the thing is: Maybe this deformed volume is much much more complicated. How complicated can it be, and how can you understand the geometry of such a deformation? This was the kind of question that really motivated me to study linear algebra. First of all, the class of linear functions $L : \mathbb{R}^3 \to \mathbb{R}^3$ gives a restricted class which can be considered to see some examples. What might $L(V)$ the set of all

[2]And let's hope it's not because some administrator decided it was "required" for your degree program.

images under a linear function $L : \mathbb{R}^3 \to \mathbb{R}^3$ look like? But really, the points is: **There should be some kind of approximation from calculus** that can be used in this situation. If I look at a single point $(x_0, y_0, z_0) \in V$, and I fix a time t, then there is a function $\phi : V \to \mathbb{R}^3$ given by $\phi(x, y, z) = F(x, y, z, t)$. This is a vector valued function of three variables, and its local behavior should be approximated by some formula involving derivatives. There is a clear "zero order formula" given by

$$\phi(x, y, z) \sim \phi(x_0, y_0, z_0),$$

but what is the **first order approximation formula**? When I understood that this formula is just like the 1-D approximation formula in the sense that it has the form

$$\phi(x, y, z) \sim \phi(x_0, y_0, z_0) + L(x - x_0, y - y_0, z - z_0)$$

where $L : \mathbb{R}^3 \to \mathbb{R}^3$ is some linear function constructed using (partial) derivatives of ϕ, then I knew

1. I needed to know how to construct this linear function L from partial derivatives, and
2. I needed to understand what the possible behaviors of such a linear function might be, i.e., how can I translate the values of the (nine) partial derivatives into geometric information about the approximate local deformation constructed using L.

These are (essentially) linear algebra questions. I need to know what linear fuctions are, how they are constructed, what properties they have, and what their images can look like. It turns out that the function L is given by matrix multiplication involving the 3×3 matrix of partial derivatives.

I'm going to stop with this discussion now, and perhaps we can come back to it during the semester as we study various properties of linear functions. Of course, this kind of discussion generally falls under the category of "applications," which is not a strong point of our textbook, but if I (or you) find the question interesting, we should not let the textook slow us down.

2 Non-Stuff

2.1 Caronavirus

If you're sick, stay home. Don't pass what you've got around to others. If you want an instructor who is hysterical concerning the scamdemic, perhaps I'm not your guy. I would hope you're not going to complain about me not wearing a mask. I hope you're not going to complain about me not being injected with experimental pharmaceuticals. A little bit of critical thinking would save you from such pitfalls.

2.2 Grades

There is a "regulation" that I give you a "grading scheme." You can find such a thing on the course page. There is a "regulation" that I assign you a grade that appears on your transcript at the end of the semester. I will probably do that. I don't really care about that. I'm not going to let it be a big part of my course. I'm not going to waste time on it, and I can't imagine why you would want to do so either. That, of course, assumes you can and have actually thought about grades. There are many different ways to think about grades. One component of "grades" is clearly pretty closely tied to the hoop-jumping I mentioned above. The actual connection between grades and thinking and learning is clearly very tenuous, and the connection with "training" is very obvious. In view of these things I suggest, we all just forget about them, more or less.

The fact remains, of course, that I will (or at least plan to) assign you a grade for your transcript at the end of the semester. I would like for you to think about that and specifically what grade you want me to assign. Here is a rough outline of how I am thinking about that:

1. If you can't live with getting a "C" on your transcript, then probably you should take another course. It's not likely to happen (statistically), but it's possible, and I wouldn't want to be responsible for giving you a grade you don't want.

2. Just about any student who takes my course and gives an honest attempt to think and learn something—just about anything—is going to get an "A." Also, any student who actually tries to think and learn—who relaxes and just gives it a try—will almost inevitably learn, and learn a lot. Such a student will get an "A." I've found that students who calculate their "grade" according to the grading scheme but actually get a "better" grade on their transcript, don't complain. Why a student would ever look at the grading scheme and calculate such a grade is something that I cannot understand—but obviously some students do it. A LOT of students do it, and I know because they write me emails and come to me at the end of the semester to tell me about it. These students still have their minds stuck deep in the matrix and can make no sense of what I've written above. Don't be one of these students.

3. I have little or no motivation to assign you a grade other than the grade you want. I've thought about this for a while, and I think there are students who don't want to just get an "A" on the transcript for the course unless they feel like they "deserve" it. These are students, of course, whose mind is to a certain extent still stuck pretty deeply in the matrix. Such a student really believes grades are serving some "greater good." That's fine. If you want a lower grade than the one you are assigned, just let me know. I'm happy to comply on that. The more common (and yes very contradictory) behavior is represented by a student who thinks grades are very important but who is incapable of thinking or learning, but still wants an "A" on his transcript. If you're such a student, you should have figured out that you should drop my course by now. I don't want to teach students like you. If you do persist, then the following discomfort is likely to take place. At the end of the semester you will have learned very little, but you probably turned in some assignments and "did something." So you get a "B" in the course on your transcript, and you're not happy about that. So you come to me and, as usual, I attempt to talk to you about linear algebra—about which you will be unable to say anything intelligent. So, you will be made uncomfortable because I will make it painfully clear that you haven't learned much of anything. I will be uncomfortable for having to do it and for having wasted my time dealing with you. (If you weren't going to try to learn linear algebra, why did you take my course?) At this point, I'll probably change your grade to an "A," which of course is consistent for me because I really don't care about the whole grade scam/sham.

4. Now you may ask: Why don't I just give all students "A" for the course grade and call it a day. I've asked myself the same question. I've even done it. I get very few complaints from students about that. There are a few who complain about other students getting grades that are better than they "deserve," and a student every now and then who complains about getting a grade he did not deserve—and wanting a "B" instead of an "A." But it's really the administrators who are rather upset about all "A" grades. Those jokers are really beyond redemption when it comes to having their heads stuck up their matrices. So practically speaking (between you, me, and the wall) it works like I've described above: The scores for assignments may be used for feedback, but don't go calculating up points and worrying about how they effect your transcript grade. Just don't do it. If you think about linear algebra, you will learn linear algebra, and get an "A." If you really somehow manage not to learn very much and get disgusted with the whole idea of thinking about the subject, you might get a "B." If you do something really strange, like at the end of the semester you didn't turn in any assignments or something like that, you'll get a "C." And hopefully everyone will be happy, or at least tolerably happy.