Final Assignment: Linear Functions (Comments)

John McCuan

April 29, 2022

As this second course in linear algebra draws to a close, I thought some comments intended to put how we have spent the semester in perspective might be in order. The course can probably be justifiably criticized from various points of view. I won't get into any of those. On the other hand, from some point of view it can be hoped that some if not many of you have gained a kind of perspective from which to view linear functions appropriate to a second course in linear algebra. To be specific the following characterizations come to mind:

First Course in Linear Algebra:

- Linear functions are determined by matrices.
  - Linear algebra is about matrices.

(I think it is likely these statements are indicative of your perception/experience.)

Second Course in Linear Algebra:

- Linear functions[1] determine matrices.
  - Linear algebra is about linear functions.

[1]on finite dimensional vector spaces

If what I have written above is accurate, the big question is (probably)

Do you understand linear functions or
are linear functions still largely a mystery?

I guess it is likely the answer is somewhat mixed or unclear. You probably understand some things, but many things are still mysterious. To illustrate or clarify (perhaps) more precisely the answer to this question, I have composed a final assignment which is a bit different from the others. I would like each student to present his or her solution of at least one problem from the final assignment in class. I do not expect (or want) any student to turn in any solutions of the final assignment problems on paper or on Canvas. Most of the problems on the final assignment involve a linear function $L : \mathbb{R}^n \to \mathbb{R}^m$ or a linear function $L : \mathbb{C}^n \to \mathbb{C}^m$ for some relatively small natural numbers n and m. Note that these are in turn examples of linear functions $L : V \to W$ where V and W are finite dimensional vector spaces. To complete most assignment problems, you should complete the "Linear Function Questionaire" on the next page. Most of the questions make good sense for any linear function $L : V \to W$, so (I suggest) such a questionaire is a good place to start in testing your understanding about linear algebra (the study of linear functions).

Linear Function Questionaire[2]

Questions that go without saying/asking:

- What are the domain and codomain of the function?
- Are the domain and codomain vector spaces?
- Is the function linear?

1. What are the null space $\mathcal{N}(L)$ and image $\mathrm{Im}(L)$?
2. What is the simplest matrix you can associate with L?
3. What words can you use to describe L?
4. What pictures can you draw to illustrate L?
5. What can you say about solving equations

$$L\mathbf{x} = \mathbf{b}$$

   where $\mathbf{b} \in \mathbb{R}^m$ is given and $\mathbf{x} \in \mathbb{R}^n$ is unknown?
6. What is the dual map L', and what significance does the dual map have?
7. Do you feel you fully understand L or are there still some mysterious aspects of L/questions you do not understand?

Given that there are only seven questions, all of which hopefully make sense to you after this course, I hope it can be said that you have made significant progress in your second course in linear algebra.

For the specific problems on the final assignment, I have also included below handy dandy classifications for certain linear functions.

[2] If you have suggestions of other important questions that should be on this questionaire, I would be very interested to hear them.

The Big Classification for Linear Functions $L : \mathbb{R}^1 \to \mathbb{R}^1$

A. $L(1) = 0$

B. $L(1) \neq 0$

- **B1.** $L(1) = 1$
- **B2.** $0 < L(1) < 1$
- **B3.** $L(1) > 1$
- **B4.** $L(1) = -1$
- **B5.** $-1 < L(1) < 0$
- **B6.** $L(1) < -1$

Overall Moral:

The Big Classification for Linear Functions $L : \mathbb{C}^1 \to \mathbb{C}^1$

A. $L(1) = 0$

B. $L(1) = a \in \mathbb{R}\backslash\{0\}$

B1. $L(1) = 1$

B2. $0 < L(1) < 1$

B3. $L(1) > 1$

B4. $L(1) = -1$

B5. $-1 < L(1) < 0$

B6. $L(1) < -1$

C. $L(1) = a + bi \in \mathbb{C}^1\backslash\mathbb{R}$

C1. $a^2 + b^2 = 1$

C1. $a^2 + b^2 < 1$

C1. $a^2 + b^2 > 1$

Hint:

$$a + bi = \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} + i\frac{b}{\sqrt{a^2 + b^2}}\right) = \sqrt{a^2 + b^2}(\cos t + i \sin t)$$

$t \in (0, 2\pi)\backslash\{\pi\}$.

The Big Classification for Linear Functions $L : \mathbb{R}^2 \to \mathbb{R}^2$

A. $L \equiv \mathbf{0}$

B. There is some $\mathbf{v} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ for which $L(\mathbf{v}) = \mathbf{0}$, and one of the following equivalent conditions holds:

(i) $L \not\equiv \mathbf{0}$.

(ii) There is some $\mathbf{w} \in \mathbb{R}^2$ for which $L\mathbf{w} \neq \mathbf{0}$.

(iii) $L\mathbf{w} \neq \mathbf{0}$ for every $\mathbf{w} \in \mathbb{R}^2 \backslash \operatorname{span}\{\mathbf{v}\}$.

(iv) *Exactly one* of the following conditions holds:

d. There is some $\tilde{\mathbf{v}} \in \mathbb{R}^2 \backslash \operatorname{span}\{\mathbf{v}\}$ such that

$$L\mathbf{w} \in \operatorname{span}\{\tilde{\mathbf{v}}\} \backslash \{\mathbf{0}\} \qquad \text{for every } \mathbf{w} \in \mathbb{R}^2 \backslash \operatorname{span}\{\mathbf{v}\}.$$

J. $L\mathbf{w} \in \operatorname{span}\{\mathbf{v}\} \backslash \{\mathbf{0}\}$ for every $\mathbf{w} \in \mathbb{R}^2 \backslash \operatorname{span}\{\mathbf{v}\}$.

B1. There is some $\tilde{\mathbf{v}} \in \mathbb{R}^2$ and some $\tilde{\lambda} \in \mathbb{R} \backslash \{0\}$ for which $(\tilde{\lambda}, \tilde{\mathbf{v}})$ is an eigenvalue/eigenvector pair.

B1.1 $\tilde{\lambda} = 1$

B1.2 $0 < \tilde{\lambda} < 1$

B1.3 $\tilde{\lambda} > 1$

B1.4 $\tilde{\lambda} = -1$

B1.5 $-1 < \tilde{\lambda} < 0$

B1.6 $\tilde{\lambda} < -1$

B2. $L\mathbf{w} \notin \operatorname{span}\{\mathbf{w}\}$ for each $\mathbf{w} \in \mathbb{R}^2 \backslash \operatorname{span}\{\mathbf{v}\}$ or alternatively/equivalently $L\mathbf{w} \in \operatorname{span}\{\mathbf{v}\} \backslash \{\mathbf{0}\}$ for every $\mathbf{w} \in \mathbb{R}^2 \backslash \operatorname{span}\{\mathbf{v}\}$.

C. $L\mathbf{w} \neq \mathbf{0}$ for every $\mathbf{w} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ and there is some $\mathbf{v} \in \mathbb{R}^2$ and some $\lambda \in \mathbb{R} \backslash \{0\}$ for which $(\lambda, \mathbf{v})$ is an eigenvalue/eigenvector pair.

C1. $\lambda = 1$

C1a There is some $\tilde{\mathbf{v}} \in \mathbb{R}^2$ and some $\tilde{\lambda} \in \mathbb{R} \backslash \{0\}$ for which $(\tilde{\lambda}, \tilde{\mathbf{v}})$ is an eigenvalue/eigenvector pair.

C1a.1 $\tilde{\lambda} = 1$

C1a.2 $0 < \tilde{\lambda} < 1$

C1a.3 $\tilde{\lambda} > 1$

C1a.4 $\tilde{\lambda} = -1$

C1a.5 $-1 < \tilde{\lambda} < 0$

C1a.6 $\tilde{\lambda} < -1$

C1b $L\mathbf{w} \notin \operatorname{span}\{\mathbf{w}\}$ for each $\mathbf{w} \in \mathbb{R}^2 \backslash \operatorname{span}\{\mathbf{v}\}$ or alternatively/equivalently $L\mathbf{w} \in \operatorname{span}\{\mathbf{v}\} \backslash \{\mathbf{0}\}$ for every $\mathbf{w} \in \mathbb{R}^2 \backslash \operatorname{span}\{\mathbf{v}\}$.

C2 $0 < \lambda < 1$

C2a There is some $\tilde{\mathbf{v}} \in \mathbb{R}^2$ and some $\tilde{\lambda} \in \mathbb{R} \backslash \{0\}$ for which $(\tilde{\lambda}, \tilde{\mathbf{v}})$ is an eigenvalue/eigenvector pair.[3]

C2a.1 $\tilde{\lambda} = 1$ $\longrightarrow$ **C1a.2**

C2a.2 $0 < \tilde{\lambda} < 1$

C2a.3 $\tilde{\lambda} > 1$

C2a.4 $\tilde{\lambda} = -1$

C2a.5 $-1 < \tilde{\lambda} < 0$

C2a.6 $\tilde{\lambda} < -1$

C2b $L\mathbf{w} \notin \operatorname{span}\{\mathbf{w}\}$ for each $\mathbf{w} \in \mathbb{R}^2 \backslash \operatorname{span}\{\mathbf{v}\}$ or alternatively/equivalently $L\mathbf{w} \in \operatorname{span}\{\mathbf{v}\} \backslash \{\mathbf{0}\}$ for every $\mathbf{w} \in \mathbb{R}^2 \backslash \operatorname{span}\{\mathbf{v}\}$.

C3 $\lambda > 1$

C3a There is some $\tilde{\mathbf{v}} \in \mathbb{R}^2$ and some $\tilde{\lambda} \in \mathbb{R} \backslash \{0\}$ for which $(\tilde{\lambda}, \tilde{\mathbf{v}})$ is an eigenvalue/eigenvector pair.

C3a.1 $\tilde{\lambda} = 1$ $\longrightarrow$ **C1a.3**

C3a.2 $0 < \tilde{\lambda} < 1$ $\longrightarrow$ **C2a.3**

C3a.3 $\tilde{\lambda} > 1$

[3] Some of the classification categories are duplicates of classification categories considered above with the roles/names of the eigenvalue/eigenvector pairs $(\lambda, \mathbf{v})$ and $(\tilde{\lambda}, \tilde{\mathbf{v}})$ reversed (as indicated).

C3a.4 $\tilde{\lambda} = -1$

C3a.5 $-1 < \tilde{\lambda} < 0$

C3a.6 $\tilde{\lambda} < -1$

C3b $L\mathbf{w} \notin \text{span}\{\mathbf{w}\}$ for each $\mathbf{w} \in \mathbb{R}^2 \setminus \text{span}\{\mathbf{v}\}$ or alternatively/equivalently $L\mathbf{w} \in \text{span}\{\mathbf{v}\} \setminus \{\mathbf{0}\}$ for every $\mathbf{w} \in \mathbb{R}^2 \setminus \text{span}\{\mathbf{v}\}$.

C4 $\lambda = -1$

C4a There is some $\tilde{\mathbf{v}} \in \mathbb{R}^2$ and some $\tilde{\lambda} \in \mathbb{R} \setminus \{0\}$ for which $(\tilde{\lambda}, \tilde{\mathbf{v}})$ is an eigenvalue/eigenvector pair.

C4a.1 $\tilde{\lambda} = 1$ $\longrightarrow$ **C1a.4**

C4a.2 $0 < \tilde{\lambda} < 1$ $\longrightarrow$ **C2a.4**

C4a.3 $\tilde{\lambda} > 1$ $\longrightarrow$ **C3a.4**

C4a.4 $\tilde{\lambda} = -1$

C4a.5 $-1 < \tilde{\lambda} < 0$

C4a.6 $\tilde{\lambda} < -1$

C4b $L\mathbf{w} \notin \text{span}\{\mathbf{w}\}$ for each $\mathbf{w} \in \mathbb{R}^2 \setminus \text{span}\{\mathbf{v}\}$ or alternatively/equivalently $L\mathbf{w} \in \text{span}\{\mathbf{v}\} \setminus \{\mathbf{0}\}$ for every $\mathbf{w} \in \mathbb{R}^2 \setminus \text{span}\{\mathbf{v}\}$.

C5 $-1 < \lambda < 0$

C5a There is some $\tilde{\mathbf{v}} \in \mathbb{R}^2$ and some $\tilde{\lambda} \in \mathbb{R} \setminus \{0\}$ for which $(\tilde{\lambda}, \tilde{\mathbf{v}})$ is an eigenvalue/eigenvector pair.

C5a.1 $\tilde{\lambda} = 1$ $\longrightarrow$ **C1a.5**

C5a.2 $0 < \tilde{\lambda} < 1$ $\longrightarrow$ **C2a.5**

C5a.3 $\tilde{\lambda} > 1$ $\longrightarrow$ **C3a.5**

C5a.4 $\tilde{\lambda} = -1$ $\longrightarrow$ **C4a.5**

C5a.5 $-1 < \tilde{\lambda} < 0$

C5a.6 $\tilde{\lambda} < -1$

C5b $L\mathbf{w} \notin \text{span}\{\mathbf{w}\}$ for each $\mathbf{w} \in \mathbb{R}^2 \setminus \text{span}\{\mathbf{v}\}$ or alternatively/equivalently $L\mathbf{w} \in \text{span}\{\mathbf{v}\} \setminus \{\mathbf{0}\}$ for every $\mathbf{w} \in \mathbb{R}^2 \setminus \text{span}\{\mathbf{v}\}$.

C6 $\lambda < -1$

C6a There is some $\tilde{\mathbf{v}} \in \mathbb{R}^2$ and some $\tilde{\lambda} \in \mathbb{R} \setminus \{0\}$ for which $(\tilde{\lambda}, \tilde{\mathbf{v}})$ is an eigenvalue/eigenvector pair.

C6a.1 $\tilde{\lambda} = 1$ $\longrightarrow$ **C1a.6**

C6a.2 $0 < \tilde{\lambda} < 1$ $\longrightarrow$ **C2a.6**

C6a.3 $\tilde{\lambda} > 1$ $\longrightarrow$ **C3a.6**

C6a.4 $\tilde{\lambda} = -1$ $\longrightarrow$ **C4a.6**

C6a.5 $-1 < \tilde{\lambda} < 0$ $\longrightarrow$ **C5a.6**

C6a.6 $\tilde{\lambda} < -1$

C6b $L\mathbf{w} \notin \mathrm{span}\{\mathbf{w}\}$ for each $\mathbf{w} \in \mathbb{R}^2 \setminus \mathrm{span}\{\mathbf{v}\}$ or alternatively/equivalently $L\mathbf{w} \in \mathrm{span}\{\mathbf{v}\} \setminus \{\mathbf{0}\}$ for every $\mathbf{w} \in \mathbb{R}^2 \setminus \mathrm{span}\{\mathbf{v}\}$.

D There is **no** $\mathbf{v} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ for which $L\mathbf{v} \in \mathrm{span}\{\mathbf{v}\}$.