Assignment 5:
Linear Functions (Section 3B)
Due Tuesday March 1, 2022

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February 22, 2022

Problem 1 *(Definitions) Let* $L : V \to W$ *be a linear function.*

(a) *Give a precise definition of the* **null space** $\mathcal{N} = \mathcal{N}(L)$ *of the linear function* $L : V \to W$*.*

(b) *Prove the null space is a vector subspace of the domain* V*.*

(c) *Give a precise definition of the* **range** $\mathcal{R} = \mathcal{R}(L)$ *of the linear function* $L : V \to W$*.*

(d) *Prove the range is a vector subspace of the codomain* W*.*

Problem 2 *(Axler 3B1) Give an example of a linear function* $L : V \to W$ *with* $\dim \mathcal{N}(L) = 3$ *and* $\dim \mathcal{R}(L) = 2$*.*

Problem 3 *(Axler 3B2) If* $L : V \to V$ *and* $T : V \to V$ *are linear functions, with*

$$\mathcal{R}(L) \subset \mathcal{N}(T),$$

Then show the compositions $(LT)^2 : V \to V$ *and* $(TL)^2 : V \to V$ *are (both) the zero map.*

Problem 4 *(Definitions) Let $f : X \to Y$ be a function.*

(a) *Give a precise definition of what it means for f to be* **surjective**.

(b) *Give a precise definition of what it means for f to be* **injective**.

(c) *Prove that a linear function $L : V \to W$ is injective if and only if $\mathcal{N}(L) = \{\mathbf{0}\}$.*

Problem 5 *(Axler 3B3) Consider a set of vectors*

$$\{v_1, v_2, \dots, v_k\}$$

in a vector space V (over a field F). Consider also the function $L : F^m \to V$ by

$$L(x_1, x_2, \dots, x_k) = \sum_{j=1}^{k} x_j \, v_j.$$

(a) *Show that L is linear, i.e., $L \in \mathcal{L}(F^n \to V)$.*

(b) *If $\{v_1, v_2, \dots, v_k\}$ is a spanning set for V, what can you say about the linear function L?*

(c) *If $\{v_1, v_2, \dots, v_k\}$ is a linearly independent set in V, what can you say about the linear function L?*

Problem 6 *(Axler 3B7) If V and W are finite dimensional vector spaces over the same field and satisfy*

$$2 \le \dim V \le \dim W,$$

then show

$$S = \{L \in \mathcal{L}(V \to W) : L \text{ is not injective}\}$$

is not a subspace of $\mathcal{L}(V \to W)$.

Problem 7 *(Axler 3B9) Let V and W be vector spaces and $L \in \mathcal{L}(V \to W)$. If*

$$\{v_1, v_2, \dots, v_k\} \text{ is a linearly independent set in } V$$

and L is injective, prove

$$\{Lv_1, Lv_2, \dots, Lv_k\} \text{ is a linearly independent set in } W.$$

Problem 8 *(Axler 3B12) Let V be a finite dimensional vector space and W and vector space. Prove that if $L : V \to W$ is a linear function, then there exists a subspace U of V such that*

(i) $U \cap \mathcal{N}(L) = \{\mathbf{0}\}$*, and*

(ii) $\mathcal{R}(L) = \{Lv : v \in U\}$.

Problem 9 *(Axler 3B13) Show that if $L : \mathbb{C}^4 \to \mathbb{C}^2$ with*

$$\mathcal{N}(L) = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1 = 5z_2 \text{ and } z_3 = 7z_4\},$$

then L is surjective.

Problem 10 *(Axler 3B31) Give an example of two linear functions*

$$L_1, L_2 \in \mathcal{L}(\mathbb{R}^5 \to \mathbb{R}^2)$$

such that

(i) $\mathcal{N}(L_1) = \mathcal{N}(L_2)$*, but*

(ii) *There is no scalar $c \in \mathbb{R}$ for which $L_2 = cL_1$.*