Assignment 2: Vector Subspaces Due Tuesday February 1, 2022

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Problem 1 *(Axler 1C3) Let* $D^2(-2,2)$ *be the collection of twice differentiable real valued functions* $f : (-2,2) \to \mathbb{R}$ *defined on the interval* $(-2,2)$.

(a) *Show that* $D^2(-2,2)$ *is a subspace of* $\mathbb{R}^{(-2,2)}$.

(b) *Show that* $V = \{f \in D^2(-2,2) : f''(-1) = 3f'(1)\}$ *is a subspace of* $D^2(-2,2)$.

(c) *Show* $W = \{f \in D^2(-2,2) : f'' = 3f'\}$ *is a subspace of* $D^2(-2,2)$.

(d) *What is* $V \cap W$*?*

Problem 2 *(Axler 1C5) In the lecture I wrote the vector space* $\mathbb{R}^2$ *in set notation as*

$$\mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}. \tag{1}$$

(a) *Write down the vector space* $\mathbb{C}^2$ *in set notation similar to (1).*

(b) *Show that* $\mathbb{R}^2 \subset \mathbb{C}^2$.

(c) *Is* $\mathbb{R}^2$ *a subspace of* $\mathbb{C}^2$*?*

Problem 3 *(Axler 1C6) Consider the sets*

$$\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\} \qquad \text{and} \qquad \{(a,b,c) \in \mathbb{C}^3 : a^3 = b^3\}.$$

(a) *Is* $\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$ *a subspace of* $\mathbb{R}^3$*?*

(b) *Is* $\{(a,b,c) \in \mathbb{C}^3 : a^3 = b^3\}$ *a subspace of* $\mathbb{C}^3$*?*

Problem 4 *(Axler 1C7) Give an example of a nonempty subset* A *of* $\mathbb{R}^2$ *satisfying*

(i) *If* $v, w \in A$*, then* $v + w \in A$*,*

(ii) *If* $v \in A$*, then* $-v \in A$*,*

but A *is not a subspace of* $\mathbb{R}^2$*.*

Problem 5 *(Axler 1C8) Give an example of a nonempty subset* B *of* $\mathbb{R}^2$ *satisfying*

If $v \in B$ *and* $a \in \mathbb{R}$*, then* $av \in B$*,*

but B *is not a subspace of* $\mathbb{R}^2$*.*

Problem 6 *(Axler 1C9) A function* $f : \mathbb{R} \to \mathbb{R}$ *with domain and codomain* $\mathbb{R}$ *is* **periodic** *if there is some positive number* p *with* $f(x+p) = f(x)$ *for all* $x \in \mathbb{R}$*. Let* P *be the set of all periodic functions. Is* P *a subspace of the collection* $\mathbb{R}^{\mathbb{R}}$ *of all functions with domain and codomain* $\mathbb{R}$*?*

Problem 7 *(Axler 1C10) Prove that if* W_1 *and* W_2 *are subspaces of a vector space* V*, then* $W_1 \cap W_2$ *is a subspace of* V*.*

Problem 8 *(sums of subspace and direct sums of subspaces)*

(a) *For each* $k = 1, 2, 3, \ldots$ *give an example of subspaces* $V_1, V_2, \ldots, V_k$ *for which*

$$V_1 + V_2 + \cdots + V_k = \bigoplus_{j=1}^{k} V_j.$$

(a) *For each* $k = 3, 4, 5, \ldots$ *give an example of subspaces* $V_1, V_2, \ldots, V_k$ *for which*

$$V_j \cap V_\ell = \{\mathbf{0}\} \qquad \text{for} \qquad j \neq \ell,$$

but

$$V_1 + V_2 + \cdots + V_k \neq \bigoplus_{j=1}^{k} V_j.$$

Problem 9 *(Axler 1C23) Prove or give a counterexample: If* W_1*,* W_2*, and* X *are subspaces of a vector space* V *satisfying*

$$V = W_1 \oplus X \qquad \text{and} \qquad V = W_2 \oplus X,$$

then $W_1 = W_2$*.*

Problem 10 *(Axler 1C24) The real vector space* $\mathbb{R}^{\mathbb{R}}$ *of all functions* $f : \mathbb{R} \to \mathbb{R}$ *with domain and codomain is the direct sum of the even functions and the odd functions:*

$$\mathbb{R}^{\mathbb{R}} = V_{\text{even}} \oplus V_{\text{odd}}$$

where

$$V_{\text{even}} = \{f \in \mathbb{R}^{\mathbb{R}} : f(-x) = f(x) \textit{ for all } x \in \mathbb{R}\}$$

and

$$V_{\text{odd}} = \{f \in \mathbb{R}^{\mathbb{R}} : f(-x) = -f(x) \textit{ for all } x \in \mathbb{R}\}.$$

Hint: Show that $[f(x) + f(-x)]/2$ *is always even.*