Assignment 12:
Eigenvalues and Eigenvectors (Axler Section 5A)
Due Tuesday April 26, 2022

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Problem 1 *(eigenvalues and eigenvectors) Given* $L : V \to V$ *linear, a field element* $\lambda \in F$ *is an* **eigenvalue** *if there is some* $v \in V\backslash\{\mathbf{0}\}$ *for which* $Lv = \lambda v$.

Given an eigenvalue $\lambda \in F$, *any vector* $v \in V\backslash\{\mathbf{0}\}$ *for which* $Lv = \lambda v$ *is called an* **eigenvector** *of* L.

Given an eigenvalue $\lambda \in F$ *and a corresponding eigenvector* $v \in V\backslash\{\mathbf{0}\}$ *with* $Lv = \lambda v$, *the pair*

$$(\lambda, v) \in F \times V\backslash\{\mathbf{0}\}$$

is called an **eigenvalue/eigenvector pair**.

(a) *Let* $(\lambda, v) \in F \times V\backslash\{\mathbf{0}\}$ *be an eigenvalue/eigenvector pair and assume* V *is finite dimensional.*

(i) *Show that if* $\mu \in F\backslash\{\lambda\}$, *then* $Lv \neq \mu v$.

Recall the identity map $\mathrm{id} = \mathrm{id}_W : W \to W$ *defined on any vector space* W *by* $\mathrm{id}(w) = w$.

(ii) *Show* $L - \lambda\,\mathrm{id} : V \to V$ *is not injective.*

(iii) *Show* $L - \lambda\,\mathrm{id} : V \to V$ *is not surjective.*

(b) *Which of the assertions* **(i)-(iii)** *of Part* **(a)** *above still hold in general even if* V *is infinite dimensional?*

Problem 2 *(eigenvalues and eigenvectors) Consider the vector space V of all* **finite sequences** *of field elements, that is, V consists of sequences*

$$\{a_n\}_{n=1}^{\infty} \subset F$$

for which there is some N such that $a_n = 0$ for all $n > N$. This vector space is a subspace of $F^{\mathbb{N}}$ and is also sometimes denoted by c_{00} with c denoting the subspace of all convergent sequences and c_0 denoting the subspace of sequences convergent to $0 \in F$. This vector space is also isomorphic to the vector space $\mathcal{P} = \mathcal{P}(F)$ of polynomials with coefficients in F.

(a) *For $j = 1, 2, 3, \ldots$, let $\mathbf{e}_j$ denote the element $\{a_n\}_{n=1}^{\infty}$ of V with $a_j = 1$ and $a_n = 0$ for $j \neq n$. Show $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots\}$ is a basis for V.*

(b) *Consider the linear function $L : V \to V$ defined by*

$$L\left(\sum_{j=1}^{k} a_j \mathbf{e}_j\right) = \sum_{j=1}^{k} a_j \mathbf{e}_{j+1}.$$

Show $L - \lambda\,\mathrm{id}$ is not surjective.

(c) *Find all eigenvalues and eigenvectors of L.*

Problem 3 *(Axler 5A7) Find all eigenvalues of $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(x, y) = (-3y, x)$.*

Problem 4 *(Axler 5A8) Find all eigenvalue/eigenvector pairs for $L : F^2 \to F^2$ by $L(w, z) = L(z, w)$.*

Problem 5 *(Axler 5A9) Find all eigenvalue/eigenvector pairs for $L : F^3 \to F^3$ by $L(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$.*

Problem 6 *(Axler 5A6) If V is a finite dimensional vector space and U is a subspace of V for which*

U is invariant with respect to **every** *linear operator $L : V \to V$,*

can you prove that either $U = \{\mathbf{0}\}$ or $U = V$?

Problem 7 *(Axler 5A14) If V is a vector space containing proper*[1] *subspaces U and W for which $V = U \oplus W$, and $L : V \to V$ is defined by*

$$L(u + w) = u \qquad \text{for } u \in U \text{ and } w \in W,$$

then find all eigenvalue/eigenvector pairs for L.

Problem 8 *(Axler 5A16) Let V be a (finite dimensional) complex vector space with bases $\mathcal{B}_1$ and $\mathcal{B}_2$. Taking the basis $\mathcal{B}_1$ for V as the domain and $\mathcal{B}_2$ as the basis for V as the co-domain, assume the matrix $A = (a_{ij})$ of a linear operator $L : V \to V$ with respect to these bases has all entries a_{ij} in the subfield $\mathbb{R} \subset \mathbb{C}$. Assume*

$$\lambda + i\mu \in \mathbb{C}$$

with $\mathrm{Re}(\lambda + i\mu) = \lambda \in \mathbb{R}$ and $\mathrm{Im}(\lambda + i\mu) = \mu \in \mathbb{R}$ is an eigenvalue for L. Show the complex conjugate $\lambda - i\mu$ is also an eigenvalue of L.

Problem 9 *(Axler 5A25-26) Let $L : V \to V$ be a linear operator.*

(a) *If the vectors v, w, and $v + w$ are eigenvectors of L, then show the eigenvalue corresponding to v is the same as the eigenvalue corresponding to w. The fact that there is a unique eigenvalue corresponding to a given eigenvector is the assertion of Problem 1 part* **(a)(i)**.

(b) *Show that if every nonzero vector $v \in V$ is an eigenvalue for L, then L is a scalar multiple of the identity operator.*

Problem 10 *(Axler 5A35-36; quotient operator) If U is a subspace of V, recall that the quotient space V/U is defined as the set of formal symbols $v + U$ where $v \in V$ with elements identified by $v + U = w + U$ when $v - w \in U$.*

(a) *Given this definition above and given a linear operator $L : V \to V$, does it make sense to define a linear function $\phi : V/U \to V/U$ by $\phi(v + U) = Lv + U$? Explain why or why not.*

(b) *We also had an alternative definition of the elements of V/U with*

$$v + U = \{v + u : u \in U\}.$$

[1] "Proper" means in this case the $U \neq \{\mathbf{0}\}$ and $U \neq V$, so the same assertions hold also for W.

With this definition, the addition and scaling in V/U are addition and scaling of sets rather than formal symbols, and these are consistent with set addition and scaling defined by

$$A + B = \{a + b : a \in A \text{ and } b \in B\} \qquad \text{and} \qquad cA = \{ca : a \in A\}.$$

Given this definition of V/U does it make sense to define a linear function $\phi : V/U \to V/U$ by $\phi(v + U) = \{Lv + Lu : u \in U\}$? Explain why or why not.

(c) *If the subspace $U \subset V$ is an invariant subspace, then we define the* **induced map on the quotient space** $\phi : V/U \to V/U$ *by*

$$\phi(v + U) = Lv + U.$$

Show the induced map ϕ is well-defined (and linear).

(d) *Assume V is finite dimensional and U is an invariant subspace of V. Prove that if λ is an eigenvalue of the induced map $\phi : V/U \to V/U$, then λ is an eigenvalue for L. Hint: Write $V = U \oplus W$ and consider the characterization given in Problem 1 above.*

(e) *Give an example to show the assumption that V is finite dimensional in the previous part is really needed in your proof. Hint: Problem 2 above.*