Assignment 11:
Duality (Section 3F) and other topics
Due Tuesday April 19, 2022

John McCuan

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Problem 1 *(real inner product space, Chapter 6) Let V be a vector space over the field $\mathbb{R}$. A function*

$$\langle \cdot\, ,\, \cdot \rangle : V \times V \to \mathbb{R}$$

is an **inner product** *if the following hold*

(i) $\langle v, w\rangle = \langle w, v\rangle$ *for all $v, w \in V$. In other words, an inner product must be* **symmetric***.*

(ii) *For each fixed $w \in V$ he function $v \mapsto \langle v, w\rangle$ is a linear function on V. This is called linearity in the first slot. In view of the symmetry condition, an inner product must be linear in both slots; that is inner product must by* **bilinear***.*

(iii) $\langle v, v\rangle \geq 0$ *for every $v \in V$ with equality if and only if $v = \mathbf{0}$. In other words, an inner product must be* **positive definite***.*

(a) *Show that if U is any subspace of an inner product space V, then*

$$U^\perp = \{v \in V : \langle v, u\rangle = 0 \text{ for all } u \in U\}$$

is a subspace of V. The space $U^\perp$ is called the **orthogonal complement** *of U or "U perp" for short.*

(b) *Show that if V is a finite dimensional inner product space and U is a* **proper** *subspace of U, in this case meaning $U \neq \{\mathbf{0}\}$ and $U \neq V$, then*

$$V = U \oplus U^\perp.$$

(c) *Show that* $U \subset U^{\perp\perp}$ *where* $U^{\perp\perp} = (U^{\perp})^{\perp}$ *is the* **double orthogonal complement** *of* U. *Note: This assertions holds even when* V *(and* U*) may be infinite dimensional.*

(d) *Show that if* V *is finite dimensional, then*

$$U^{\perp\perp} = U.$$

Note: This also holds if U *is finite dimensional, even if* V *is infinite dimensional, and more generally, this holds if* U *is* **closed** *even if* U *is infinite dimensional, but we have not talked about what it means for* U *to be closed.*

Problem 2 *(real normed space) Let* V *be a vector space over the field* $\mathbb{R}$. *A function*

$$\| \cdot \| : V \to [0, \infty)$$

is a **norm** *if the following hold*

(i) $\|cv\| = |c| \, \|v\|$ *for every* $c \in \mathbb{R}$ *and* $v \in V$. *In other words, a norm must be* **non-negative homogeneous**.

(ii) $\|v + w\| \leq \|v\| + \|w\|$ *for every* $v, w \in V$. *This is called the* **triangle inequality** *for the norm.*

(iii) $\|v\| = 0$ *if and only if* $v = \mathbf{0}$. *In other words, a norm must be* **positive definite**.

(a) *Given a real inner product space* V, *show that* V *is also a real normed space with*

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

Note: This norm is called the **norm induced by the inner product** *or the* **induced norm**.

Caution/Hint: Showing **(i)** *and* **(iii)** *should be relatively straightforward, but showing the triangle inequality for the norm may be quite difficult for you if you haven't seen it before; you may want to look up the proof in a book or on the internet.*

(b) *Show that* $\mathbb{R}^n$ *is a real inner product space (and hence a real normed space) with*

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{j=1}^{n} v_j w_j.$$

This inner product is called the **standard dot product** *on* $\mathbb{R}^n$.

(c) *Let $p_1, p_2, \ldots, p_n$ be positive real numbers. Show that*

$$\langle \mathbf{x}, \mathbf{y} \rangle_* = \sum_{j=1}^{n} p_j x_j y_j$$

defines an inner product on $\mathbb{R}^n$.

(d) *In the case $(p_1, p_2, \ldots, p_n) \neq (1, 1, \ldots, 1) \in \mathbb{R}^n$, the inner product in part (c) is a* **non-standard inner product***. Take $n = 2$ and draw the unit circle*

$$\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_* = 1\}$$

determined by the norm induced by a non-standard inner product.

Problem 3 *(generalized bases and dimension) Recall that if V is a finite dimensional vector space, then a basis $\{v_1, v_2, \ldots, v_k\}$ of V is defined to be a finite subset of V which is* **linearly independent** *and is a* **spanning set***. With this definition, the* **dimension** *of a finite dimensional vector space is defined to be the number of elements in a basis. These constructions can be generalized as follows:*

Given a vector space X, **any** *subset $A \subset X$ is*

(i) *a* **spanning set** *if every vector $x \in X$ can be written as a (finite) linear combination*

$$x = \sum_{j=1}^{k} a_j x_j$$

for some vectors $x_1, x_2, \ldots, x_k \in A$.

(ii) **linearly independent** *if every (finite) linear combination*

$$\sum_{j=1}^{k} a_j x_j$$

for some vectors (distinct) $x_1, x_2, \ldots, x_k \in A$ for which

$$\sum_{j=1}^{k} a_j x_j = 0$$

must have $a_1 = a_2 = \cdots = a_k = 0$.

Note: *These definitions do not require the set A to be a finite set.*

Given a vector space X, **any** *subset $B \subset X$ is a* **basis** *if B is a linearly independent spanning set.*

(a) *Show that every generalized basis B for a finite dimensional vector space V is a basis according to the old (finite dimensional) definition of basis.*

(b) *Show that every basis $\{v_1, v_2, \ldots, v_k\}$ of a finite dimensional vector space V is a basis in the generalized sense.*

(c) *Show that if B is a (generalized) basis for a vector space X, then every vector $x \in X$ is written* **uniquely** *as a linear combination*

$$x = \sum_{j=1}^{k} a_j x_j$$

for some (distinct) vectors $x_1, x_2, \ldots, x_k \in B$ where the uniqueness holds up to a reordering of the basis vectors. Hint: Use induction.

Recall that for a finite dimensional vector space V the dual space V' is isomorphic to V. The objective of the next three problems is to suggest how one would go about showing X' is **not isomorphic** *to X when X is infinite dimensional. There are more interesting things in these problems as well.*

Problem 4 *(polynomial basis) Remember*

$$\mathcal{P} = \mathcal{P}(F) = \left\{ \sum_{j=0}^{k} a_j z^j : k \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \text{ and } a_j \in F \textit{ for } j = 0, 1, 2, \ldots, k \right\}$$

is the vector space of polynomials with coefficients in a field F.

(a) *Find a basis for the vector space $\mathcal{P} = \mathcal{P}(F)$ of polynomials with coefficients in a field.*

(b) *Show the dual space $\mathcal{P}'$ is isomorphic to $F^{\mathbb{N}}$ the vector space of sequences $\{a_j\}_{j=1}^{\infty}$ in F.*

Problem 5 *(ordinals) You are familiar with some numbers and some sets of numbers. For example* $\mathbb{N} = \{1, 2, 3, \ldots\}$ *is the set of* **natural numbers***, and* $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ *is the set of* **natural numbers with zero**. $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \ldots\}$ *is the set of* **integers**, $\mathbb{Q} = \{p/q : p \in \mathbb{Z} \text{ and } q \in \mathbb{N}\}$ *is the set of rational numbers, and* $\mathbb{R}$ *is the set of* **real numbers**. *Some of the numbers in these sets are* **ordinal numbers** *(and some are not). There are also other ordinal numbers you may not know about. Here is an introduction:*

(i) *Ordinal numbers have to do with* **ordering**. *In order to talk about ordering, we* **reinterpret** *numbers as sets and say two ordinal numbers a and b satisfy*

$$a < b \qquad \text{if} \qquad a \subset b.$$

This, of course, seems strange, but it's also rather natural when you get used to it.

(ii) *As a set the number 0 is the first ordinal and it* **is** *the empty set. You'll notice* $0 = \phi \in \mathbb{N}_0 \cap \mathbb{Z}$. *Thus, 0 and ϕ are two different names for the same thing: the first ordinal number.*

(iii) *The second ordinal number is* $1 = \{0\} = \{\phi\}$. *Notice that $0 < 1$ (by definition) because* $0 = \phi \subset 1 = \{\phi\}$. *In fact, $0 = \phi$ is a subset of every ordinal number because there are no elements in 0.*

(iv) *As a set/ordinal* $2 = \{0, 1\}$, *and*

$$k + 1 = \{0, 1, 2, \ldots, k\}.$$

Thus, all the numbers $0, 1, 2, 3, \ldots$ in $\mathbb{N}_0$ are ordinals and they are ordered in the way you would imagine. These are called the finite ordinals.

(v) *Remember that ordering is important for ordinals. Each ordinal is an* **ordered set**. *This means one can compare every two (distinct) elements a and b in an ordinal ν and either $a < b$ or $b < a$. A bijection $f : \nu \to \mu$ between ordinals is said to be* **order preserving** *if $f(a) < f(b)$ whenever $a < b$.*

(a) *Prove (or at least try to prove or convince yourself) that there does not exist a bijection between n and m for any (distinct) $n, m \in \mathbb{N}_0$.*

(b) *Prove (or at least try to prove or convince yourself) that given any finite set A there exists exactly one $n \in \mathbb{N}_0$ for which there exists a bijection $f : A \to n$.*

(c) *The first ordinal you may not know about is ω. This ordinal is the smallest* **infinite ordinal***:*

$$\omega = \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$$

with the obvious ordering. Congratulations: You probably just learned how to count higher than you ever have before! Show (or at least try to show or convince yourself) that there does not exist a bijection $f : n \to \omega$ for any finite ordinal $n \in \mathbb{N}_0$. In fact, $n < \omega$ for every $n \in \mathbb{N}_0$.

(d) *The next infinite ordinal is $\omega + 1 = \mathbb{N} \cup \{\omega\}$ with the ordering determined by the condition $n < \omega$ for all $n \in \mathbb{N}$. Show that $\omega < \omega + 1$.*

(e) *Show there exists a bijection $f : \omega \to \omega + 1$.*

(f) *Show there does* **not** *exist an order preserving bijection $f : \omega \to \omega + 1$.*

Problem 6 *(cardinality and cardinals) You can probably count much higher than you could before the previous problem:*

$$0, 1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \ldots$$

with $\omega + k + 1 = [\omega + k] \cup \{\omega + k\}$. It is, of course, possible to count even higher if you know how:

$$2\omega = \cup_{k=0}^{\infty}(\omega + k).$$

(a) *If α and β are among the infinite ordinal numbers mentioned above,*[1] *then there exists a bijection $f : \alpha \to \beta$.*

(b) *Any two sets A and B for which there is a bijection $f : A \to B$ are said to have the same* **cardinality***, or* **number of elements***. If this is the case, we write*

$$\#A = \#B,$$

and we say $\#A$ is the cardinality of A. Thus, the previous part of this problem asserts that all the infinite ordinal numbers mentioned so far have the same cardinality. Also, part (b) of Problem 5 asserts that every finite set has the cardinality of precisely one finite ordinal. Prove that 2^ω, the set of all functions from ω to $2 = \{0, 1\}$ does not have the cardinality of any ordinal mentioned

[1] Just to be clear, there are other infinite ordinals coming, but these have not been mentioned above. The ones "mentioned above" at this point are ω, $\omega + k$ for $k \in \mathbb{N}$, and 2ω.

above. Hint: Assume there is a bijection $f : \mathbb{N} \to 2^\omega$ so that every function $\phi : \mathbb{N}_0 \to \{0, 1\}$ can be found in a sequence

$$\phi_1, \phi_2, \phi_3, \ldots.$$

Then construct some other $\phi : \mathbb{N}_0 \to \{0, 1\}$ which is not in the sequence. Thus, f cannot be surjective.

(c) *The previous part of this problem suggests that the set 2^ω is rather bigger (in terms of cardinality) than any of the infinite ordinals mentioned above. Are you ready to learn to count higher? Yes, 2^ω is also an ordinal, but it is a rather different kind of ordinal: Let $\mathcal{O}$ denote the ordinal numbers, then*

$$\mathcal{O} = \mathbb{N}_0 \cup \mathcal{C} \cup \mathcal{U}$$

where

$$\mathcal{C} \supset \{\omega, \omega + 1, \omega + 2, \ldots, 2\omega, 2\omega + 1, \ldots\}$$

is the set of all ordinals ν for which there is a bijection $f : \mathbb{N} \to \nu$, and $\mathcal{U}$ is the set of all larger ordinals.

Definition 1 *A set A for which there exists a bijection $f : A \to n$ for some ordinal $n \in \mathbb{N}_0$ is said to have* **finite cardinality** *or just simply to be a finite set. A set A for which A is a finite set* **or** *there exists a bijection $f : \mathbb{N} \to A$, i.e., A has the same cardinality as any of the*

$$\omega, \omega + 1, \omega + 2, \ldots, 2\omega, 2\omega + 1, \ldots \in \mathcal{C}$$

is said to be **countable**. *An ordinal in $\mathcal{U}$, like 2^ω, $2^\omega + 1$, $2^{\omega+1}$ or 3^ω, is said to be* **uncountable**.

Which is the bigger ordinal $2^{\omega+1}$ or 3^ω (or are they the same)?

(d) *Here is something interesting:*

$$\Omega = \bigcap_{\nu \in \mathcal{U}} \nu$$

is an uncountable ordinal called the **first uncountable ordinal** *or the smallest uncountable ordinal. We can also choose a specific given set, say an ordinal, to* **represent the cardinality** *of any set. Of course, this requires one to be able to count the elements in an arbitrary set, which is quite an impressive thing* *to be able to do. These representative ordinals are called* **cardinal numbers**. *The cardinal numbers only take account of cardinality but not ordering. We can (and must) take $n \in \mathbb{N}_0$ as cardinals for the finite sets. We can choose any element of $\mathcal{C}$ to represent the cardinality of $\mathbb{N}$ and $\mathbb{N}_0$ and ω. We usually take $\omega = \mathbb{N}_0$, but whatever set one chooses this cardinality (as a cardinal number) is denoted $\aleph_0$.*

> *$\aleph_0$ is representative of the cardinality of every infinite countable ordinal (and set).*

While cardinal numbers do not take account of ordering by nature, there is also an ordering on the cardinal numbers themselves: We say the cardinal numbers γ and η satisfy $\gamma < \eta$ if there is an injection $f : \gamma \to \eta$ but no injection $g : \eta \to \gamma$. There is also **cardinal arithmetic** *rather like the* **ordinal arithmetic** *one uses to count.*

The cardinality of the first uncountable ordinal Ω is called $\aleph_1$. This cardinality is **not** *representative of the cardinalities of every uncountable ordinal in $\mathcal{U}$. Show $\aleph_1 \le \#\mathbb{R} = 2^{\aleph_0}$. Hint: Binary decimal representation.*

It turns out that one has a strange choice at this point: One can either assume

$$\aleph_1 < \#\mathbb{R} = 2^{\aleph_0} \qquad \text{or} \qquad \aleph_1 = \#\mathbb{R} = 2^{\aleph_0}.$$

The latter assumption is called the **continuum hypothesis**. *The continuum hypothesis was stated as a conjecture by Georg Cantor in 1878. Paul Cohen received the fields medal in 1966 for his 1963 proof that either assumption is possible/consistent. Cohen's result is called the* **independence of the continuum hypothesis**.

Note: *We have not established the assertion above that there are uncountable ordinals with cardinality greater than $\aleph_1$. Of course, we can choose for this to be the case (according to Paul Cohen). It can also be proved that $\aleph_1 < 2^{\aleph_1}$, which is rather easier than reading Cohen's proof.*

Problem 7 *(infinite dimensions) Let X be a vector space with a (generalized) basis B. The* **dimension** *of X is then defined to be $\#B$, the cardinality of B.*

(a) *Prove that if X is a vector space with a basis B, then every basis for X has the same cardinality as B. This means the notion of dimension is well-defined.*

(b) *Prove that the dual space $\mathcal{P}'$ of the vector space of polynomials $\mathcal{P}$ considered in Problem 4 above is* **not isomorphic** *to $\mathcal{P}$.*

Problem 8 *(Axler 3F30) If V is a finite dimensional vector space and $\{\phi_1, \phi_2, \dots, \phi_k\}$ is a linear independent set in the dual space V', then show*

$$\dim\left(\cap_{j=1}^{k} \mathcal{N}(\phi_j)\right) = \dim V - k.$$

Note: The dual vectors $\phi_1, \phi_2, \dots, \phi_k$ are not intended to be any particular "standard" dual basis vectors.

Problem 9 *(Axler 4.5) If $k \in \mathbb{N}$ and $a_1, a_2, \dots, a_k$ are distinct field elements and $b_1, b_2, \dots, b_k$ are any field elements, then prove there exists a unique polynomial $p \in \mathcal{P}_m(F)$ for which*

$$p(a_j) = b_j \qquad \text{for} \qquad j = 1, 2, \dots, k.$$

Problem 10 *(Axler 5A1-3) Let $L : V \to V$ be a linear function (what Axler calls an* **operator***). A subspace $U \subset V$ is called* **invariant** *if the restriction*

$$M = L\big|_U : U \to V$$

has image

$$\text{Im}\left(L\big|_U\right)$$

satisfying

$$\text{Im}\left(L\big|_U\right) \subset U.$$

In this case, we can consider the restriction $M : U \to U$ as an operator on U.

(a) *Show any subspace of $\mathcal{N}(L)$ is invariant.*

(b) *Show that if U is a subspace for which $\text{Im}(L) \subset U$, then U is invariant.*

(c) *Show that if $S, T \in \mathcal{L}(V \to V)$ with $ST = TS$, then $\mathcal{N}(S)$ and $\text{Im}(S)$ are invariant subspaces for T.*

(d) *Define what it means for $v \in V$ to be an* **eigenvector** *of L, and prove that if U is a one-dimensional subspace of V, then there exists an eigenvector for L.*