Assignment 10:
Duality (Sections 3F)
Due Tuesday April 12, 2022

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Problem 1 *(definition) Let* V *be a vector space over a field* F.

(a) *State carefully the definition of the* **dual space** V' *of* V.

(b) *Prove that the dual space* V' *is a vector space. Hint: It is a subspace of a known vector space.*

(c) *Prove* $\dim V' = \dim V$ *(when* V *is finite dimensional).*

Problem 2 *(definition) Let* V *and* W *be vector spaces and* $L \in \mathcal{L}(V \to W)$ *be a linear function.*

(a) *Define carefully the* **dual map** L' *of* L.

(b) *If* $M \in \mathcal{L}(V \to W)$ *is another linear function, show* $(L + M)' = L' + M'$.

(c) *If* $a \in F$, *show* $(aL)' = aL'$.

Problem 3 *(double dual) Let* V *be a finite dimensional vector space,* V' *the dual space of* V, *and* V'' *the dual space of* V'. *Consider a function* $M : V \to V''$ *by*

$$Mv(\phi) = \phi(v).$$

(a) *Show* M *is well-defined.*

(b) *Show* M *is linear.*

(c) *Show M is an isomorphism.*

Problem 4 *(definition) Let V be a vector space and let S be a subset of V. We define the* **annihilator** *of S to be*

$$\mathcal{A}(S) = \{\phi \in V' : \phi(v) = 0 \text{ for all } v \in S\}.$$

(a) *Show $\mathcal{A}(S)$ is a subspace of V'.*

(b) *Notice that we defined the annihilator for general subsets of a vector space rather than for subspaces. Show that there is a subspace W of V such that*

$$\mathcal{A}(W) = \mathcal{A}(S).$$

(c) *Is the subspace W in part* **(b)** *of this problem unique?*

Problem 5 *(Theorem 3.106) Let V be a vector space and W a subspace of V. Define the* **injection** *of W into V as the function $i : W \to V$ by $i(w) = w$.*

(a) *Show the injection is linear and injective.*

(b) *Prove*

$$\mathcal{A}(W) = \mathcal{N}(i')$$

where $i' : V' \to W'$ is the dual map of i.

Problem 6 *(Axler 3F3) Let V be a finite dimensional vector space, and assume $v \in V \backslash \{\mathbf{0}\}$. Prove there exists some $\phi \in V'$ with $\phi(v) = 1$.*

Problem 7 *(Axler 3F4) Let V be a finite dimensional vector space with a* **proper** *subspace W.* **Proper** *here means $W \neq V$. Prove there exists some $\phi \in V' \backslash \{\mathbf{0}'\}$ such that*

$$\phi\big|_W \equiv 0.$$

Problem 8 *(Axler 3F7) Let k be a positive integer and consider the vector space $\mathcal{P}_k$ of polynomials of degree at most k. Given the basis*

$$\{1, x, x^2, \ldots, x^k\}$$

for $\mathcal{P}_k$, show the **dual basis** *for $\mathcal{P}_k'$ is given by $\{\phi_0, \phi_1, \ldots, \phi_k\}$ where*

$$\phi_j[p] = \frac{1}{j!}\, p^{(j)}(0) \qquad \text{for } p \in \mathcal{P}_k \text{ and } j = 0, 1, 2, \ldots, k.$$

Problem 9 *(Axler 3F9) Let V be a vector space with basis $\{v_1, v_2, \ldots, v_n\}$ and let $\{\phi_1, \phi_2, \ldots, \phi_n\}$ be the corresponding dual basis for V'. Show that*

$$\phi = \sum_{j=1}^{n} \psi(v_j)\ \phi_j \qquad \textit{for } \phi \in V'.$$

Problem 10 *(Axler 3F11) Let $A = (a_{ij})$ be a nonzero $m \times n$ matrix.*

(a) *State carefully the definition of the* **column rank** *of A.*

(b) *State carefully the definition of the* **row rank** *of A.*

(c) *If there exist vectors $(a_1, a_2, \ldots, a_m) \in F^m$ and $(b_1, b_2, \ldots, b_n) \in F^n$ such that*

$$a_{ij} = a_i b_j,$$

then what is the column rank of A?

(d) *If there exist vectors $(a_1, a_2, \ldots, a_m) \in F^m$ and $(b_1, b_2, \ldots, b_n) \in F^n$ such that*

$$a_{ij} = a_i b_j,$$

then what is the row rank of A?

(e) *If the column rank of A is 1, then show there exist vectors $(a_1, a_2, \ldots, a_m) \in F^m$ and $(b_1, b_2, \ldots, b_n) \in F^n$ such that*

$$a_{ij} = a_i b_j.$$

(f) *If the row rank of A is 1, then show there exist vectors $(a_1, a_2, \ldots, a_m) \in F^m$ and $(b_1, b_2, \ldots, b_n) \in F^n$ such that*

$$a_{ij} = a_i b_j.$$