Assignment 1: Complex Numbers and Vector Spaces Due Tuesday January 25, 2022

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Problem 1 *(Axler 1A1) Given real numbers* a *and* b *find/write*

$$\frac{1}{a+bi}$$

in the form $x + iy$ *for real numbers* x *and* y.

Problem 2 *(Axler 1A3) Find the square roots of* i.

Problem 3 *Consider the subspace*

$$V = \{t(1+i, 2-i) : t \in \mathbb{C}\}$$

in $\mathbb{C}^2$. *Is there a point* $(3, x)$ *in* V*?*

Problem 4 *(Axler 1A11) Consider the subspace*

$$V = \{t(2-3i, 5+4i, -6+7i) : t \in \mathbb{C}\}$$

in $\mathbb{C}^3$. *Is the point* $(12-5i, 7+22i, -32-9i)$ *in* V*?*

Problem 5 *At the bottom of page 10 Axler notes that a general* **field** F *is a set containing elements* 0 *and* 1 *and satisfying the properties listed in the gray box on page 3. More precisely, there are two operations (multiplication and addition) with*

1. $a + b = b + a$ *and* $ab = ba$ *for all* $a, b \in F$.
2. $(a + b) + c = a + (b + c)$ *and* $(ab)c = a(bc)$ *for all* $a, b, c \in F$.
3. $a + 0 = 0$ *and* $1a = a$ *for all* $a \in F$.
4. *For every* $a \in F$ *there is some* $b \in F$ *with* $a + b = 0$.
5. *For every* $a \in F\backslash\{0\}$ *there is an element* $b \in F$ *with* $ab = 1$.
6. $a(b + c) = ab + ac$ *for all* $a, b, c \in F$.

Look up the names of these properties on the top of page 3 and figure out which property you need to prove the following:

> *Every field* F *contains an element* -1 *such that* $(-1)a = -a$ *for every* $a \in F$.

(And prove it.)

Problem 6 *(Axler 1B1) Prove that* $-(-v) = v$ *for every vector* v *in a vector space* V *over a field* F.

Problem 7 *(Axler 1B2) Prove that if* $av = 0$ *for* $a \in F$ *and* $v \in V$ *(where* F *is a field and* V *is a vector space over* F*), then* $a = 0$ *or* $v = 0$. *Note: The single symbol* 0 *is used to denote two different things here, namely the additive identity* $0 \in F$ *in the field and* $0 \in V$ *the additive identity in the vector space. Note also that what the symbol is denoting can always be indicated by writing the set to which it belongs, as I have just done. Every time you write* 0 *in your statement/proof/discussion, make sure to indicate which zero you are writing.*

Problem 8 *(Definition 1.19 page 12) Show there can be only one additive inverse in a vector space over a field.*

Problem 9 *(Vector space of functions 1.23 page 14) Let* S *be any set and denote the set of complex valued functions on* S *by* $V = \mathbb{C}^S$. *Define operations on* V *and verify that these make* V *a complex vector space.*

Problem 10 *(Axler 1B5) Let W be a set and F be a field with operations of addition in W and scaling of elements of W by elements of F defined such that the following properties hold*

1. $u + v = v + u$ *for all* $u, v \in W$.
2. $(u + v) + w = u + (v + w)$ *and* $(ab)v = a(bv)$ *for all* $u, v, w \in W$ *and* $a, b \in F$.
3. *There is an element* $0 \in W$ *for which* $v + 0 = v$ *for every* $v \in W$.
4. *The scaling* $0v$ *(with* $0 \in F$*) is the element* $0 \in W$ *for every* $v \in W$.
5. *The scaling* $1v$ *(with* $1 \in F$*) is the element* $v \in W$ *for every* $v \in W$.
6. $a(u + v) = au + av$ *and* $(a + b)v = av + bv$ *for all* $a, b \in F$ *and* $u, v \in W$.

Prove that for each $v \in W$*, there exists an element* $w \in W$ *with* $v + w = 0$*. Note carefully which properties of* W *you need to prove this and which ones can be ignored/omitted.*