MIT Probability Problem 1

This is a problem having to do with **poker hands**. In the statement of the problem a poker hand called **one-pair** is described as "more common" than two other poker hands. These other hands are called **two-pair** and **three-of-a-kind**, and these hands are described in detail. One is asked to determine the **probability** of these two hands and to determine which is "more likely."

I will try to describe all the terms mentioned in the problem including answers to the following questions:

What is a **deck** of cards?
What are the **rank** and **suit** of a card?
How can symbols be used to mathematically model cards?
What is a (poker) **hand**?
What are the various named hands (**one pair**, **two pair**, **three of a kind**, etc.) in poker?
How can you count the numbers of various hands?
What does one mean by the **probability** of a certain hand in poker?

I will also attempt to describe some general counting techniques including the meaning and some uses of **permutations** and **combinations**. I will use some elementary terminology and notation from **set theory** including the idea of **cardinality** with the **cardinality**, or **number of elements**, of a set S denoted by

$$\#S$$

You may need to read a little bit about the set theory elsewhere, but probably everything related to sets used below will be "self-explanatory."

Background

Let us assume there are 52 cards in a "standard deck" of cards. More precisely, there are four "aces" with an "A" on each one. An ace is considered to have **rank** 1 (or sometimes rank 14 if "counted high"). In harmony with having **rank** 1, one of the aces has a single **club** on it. A **club** looks like this:

♣

There are three other aces, each with a different symbol called the **suit** of the card. The other **suits** or other **suit symbols** are **diamonds**, **hearts**, and **spades**. The other **suit symbols** look like this:

♢ ♡ ♠

Similarly, there are four cards of rank 2. The “two of clubs” has two club symbols, the “two of diamonds” has two diamond symbols on it, and so on. This continues for each natural number up to 10 so that there are 40 **numbered cards** in the deck. For example, the “seven of spades” has on it seven spade symbols, and we can denote it symbolically by

7♠

The four aces can be denoted by

1♣ 1♢ 1♡ 1♠

or

A♣ A♢ A♡ A♠

There are 12 more cards called “face cards” having ranks 11, 12, and 13 and called “jacks,” “queens,” and “kings” respectively (with four in each suit). We can use the same convention for the symbolic representation of face cards so that the queen of diamonds is denoted by

12♢ or 12♢

More often we will use the notation

Q♢ (J♣ K♡ etc.)

In general, a rank together with a suit may be called a **card symbol**.

A **poker hand** of cards is a collection of 5 cards from the deck. The collection of cards in a hand is considered unordered, so there is no difference between

3♡ 6♠ K♣ K♢ A♡

and

K♣ K♢ 1♡ 3♡ 6♠

I want to start by counting the number of (all) poker hands. In order to do this, I am going to first do something unusual: I am going to consider **poker hands distinguished by order**. You may imagine the cards in the poker hand being handed out, or **dealt**, and accumulated one by one, and I wish to keep track of which card was dealt first, second, third and so on to the fifth. For example, in the hand above perhaps the ace of hearts was dealt first, and then the king of diamonds, and so on until the three of hearts was dealt last. I want to consider this quite different from the hand in which the six of spades was dealt first, the three of hearts second, and so on until the king of clubs was dealt last.

Mathematically, I can model a deck of cards with a **set** containing (all) the card symbols.

C = { 1♣, 1♢, 1♡, 1♠, 2♣, 2♢, 2♡, 2♠, ..., K♣, K♢, K♡, K♠ }

The hands of cards correspond to, or are modeled by, the collection of subsets of the set C containing five elements. The unusual set I have in mind is somewhat simpler consisting of all **ordered 5-tuples** of distinct elements from C.

In order to count the number of elements in such a set, let us consider an even simpler example, namely the collection of **ordered pairs** of distinct elements from C corresponding to **blackjack hands distinguished by order**. Blackjack hands consist of only two cards. For a set of “two piece” objects like our cards determined by a rank and a suit, the total number of objects is the product of the number of possible first pieces (13 ranks) and the number of possible second pieces (4 suits). This is one way to see there are 52 cards in the deck. Similarly, ordered blackjack hands consist of two pieces, but the two pieces are of the
same kind (both cards) and cannot be the same card. In order to count such “two piece” objects we need a slightly different multiplication principle of counting:

If a set T consists of all “two piece” objects (a,b) with the pieces a and b coming from the same set S with n elements but with a and b distinct, then there are n(n-1) ordered “two piece” objects in T.

For a hand in blackjack the first card may be dealt in 52 ways, and the second card can be dealt in 51 ways to complete/construct the (ordered) hand. Thus, there are 52 (51) = 2652 blackjack hands distinguished by order. That is,

#(C^2\backslash {(a,a): a ∈ C}) = 2652.

The set D = {(a,a): a ∈ C} is called the **diagonal** of C^2. The ordered pairs in the diagonal can not, of course, correspond to ordered hands because there is only one of each card in a standard deck.

The counting principle above can be extended to determine the number of hands with three cards distinguished by order. Card games involving three card hands are palace (also known as shithead) and three-card monte. As noted above, the first two cards can be dealt in 2652 distinct ways, so an entire (ordered) three card hand can be completed/constructed with a “second” step of dealing the last card in 50 ways for a total of 52 (51) (50) = 132600 ordered hands.

In terms of sets, the subset of ordered triples in C^3 corresponding to possible ordered hands

K = { (c_1,c_2,c_3) ∈ C^3 : c_1 ≠ c_2, c_1 ≠ c_3, c_2 ≠ c_3 }

is a little more complicated to express.

Continuing, we obtain the number of poker hands distinguished by order, namely

52 (51) (50) (49) (48) = 311875200.

That is, three hundred and eleven million, eight hundred seventy-five thousand, and two hundred.

Exercise: Formulate a principle of counting for finding the number of distinct ordered “k piece” objects with each piece coming from a fixed set with n elements if no two of the pieces can be the same.

Exercise: Describe/formulate a counting problem in which your counting principle leads to the construction of n! = n (n-1) (n-2) ⋯ (2) (1) possible objects.

Each of the numbers (of hands distinguished by order) we have calculated above may be expressed in terms of the mathematical **factorial function** which gives the product of a natural number n with all the natural numbers less than n:

n! = n (n-1) (n-2) ⋯ (2) (1).

For ordered blackjack hands we have

2652 = 52!/(52-2)!

For ordered three-card hands,

132600 = 52!/(52-3)!

and in general the number of ordered hands with n cards taken from a standard deck of 52 cards (assuming n is less than 53) is

52!/(52-n)!

You may recall, or note, here that we need to use 0! = 1 in the case when n = 52. Note this special case illustrates something very clearly: There are more hands where order is distinguished (52! in this case) than there are regular hands of cards. There is only one hand of 52 cards (consisting of the entire deck).

Exercise: If you have a deck of m cards and $n < m$, then how many hands distinguished by order are possible?

Exercise: Technically three-card monte is not “played” with a standard deck of 52 cards but rather with a deck of only 3 cards. Order, however, definitely matters. Calculate how many three-card monte hands are possible.

Before we proceed to count regular hands, I want to introduce some terminology and notation associated with the number

$$m!/(m-n)!$$

This number is called the **permutation** of m (things) taken n (at a time). It is, as we have seen, the number of distinct n-tuples with entries taken from among m distinct objects (no two of which can appear more than once). The **permutation function** is a function of two integers:

$$P(m,n) = m!/(m-n)!$$

and is sometimes denoted with various super- and sub-scripts, perhaps the most common being

$${}_{m}P_{n}$$

A corresponding terminological description for the same number is **m permute n**.

In summary, there are

$$52\ (51)\ (50)\ (49)\ (48) = 52!/(52-5)! = P(52,5) = 311875200$$

ordered poker hands. This number is larger than the number of poker hands, where order does not matter, and we want to be able to determine how much larger.

I return to the simpler situations of blackjack and palace. Notice that in blackjack, each of the two cards must be distinct. Therefore, each ordered pair (a,b) in C^2 has exactly one corresponding ordered pair (b,a) in C^2 which corresponds to exactly the same hand modeled by the (sub)set { a, b }. This means when counting ordered pairs, we have counted exactly twice the number of unordered subsets. If we want blackjack hands, then P(52,2) is off by a factor of 2, and actual number of blackjack hands is

52!/[(52-2)! 2] = 52 (51)/2 = 1326.

There is a slightly different way to look at this. That factor of 2 is precisely the number of distinct ordered pairs obtained by permuting two specific cards (once those cards are specified). That is, once we decide on a hand modeled by the (unordered) set {a,b}, then we know there are P(2,2) = 2!/(2-2)! = 2 ordered pairs that will be counted in the hands distinguished by order, namely (a,b) and (b,a).

This point of view extends to hands containing more cards. How many ordered triples will correspond to the single palace hand { a, b, c }? The answer is P(3,3) = 3! = 6. This tells us that the number of three card palace hands is

52 (51) (50)/6 = P(52,3)/3! = 22100,

and the number of hands consisting of 52 cards is

P(52,52)/P(52,52) = 1.

Finally then we can easily calculate the number of poker hands:

52 (51) (50) (49) (48)/5! = 52 (51) (50) (49) (48)/120 = 2598960,

somewhere in the neighborhood of two-and-a-half million (or about 2.6 million to be a little more exact).

This is a nice number to have. Before we go on to consider various specific hands and sets of hands, I want to introduce some terminology and notation associated with the number(s) of hands discussed above.

The number of subsets of a given set of m (distinct) elements consisting of n elements, that is the cardinality of the set of subsets h of a set C with #C = m and #h = n, is

C(m,n) = P(m,n)/P(n,n) = P(m,n)/n!.

This number is called the **combination** of m (things) taken n (at a time). For a combination, order does not matter, and this is the same as the number of n card hands possible when using a deck of m cards. This number is also the number of ways to select n objects from among a collection of m objects in cases where order does not matter and is sometimes referred to simply as “m choose n.” The combination C(m,n) is also denoted by

$$C_{m\,n} \quad \text{and} \quad \binom{m}{n}.$$

These numbers are also known as **binomial coefficients**.

Exercise: Express/expand (a+b)^m in terms of binomial coefficients. Hint: Use (mathematical) induction and some properties/identities involving combinations.

Exercise: Recall Pascal’s triangle and express the entries in terms of binomial coefficients.

One pair in poker

By a **one pair** hand in poker we mean a hand in which two cards, and exactly two cards (out of the five) have the same rank. We can use the multiplication principle, the extended multiplication principle (for objects constructed in some finite number of steps), and the counting techniques involving permutations and combinations to calculate/count the number of one pair hands.

Before we begin, it may be useful to state/describe all the conditions considered to constitute a specified hand in poker.

Specified hands

1. Royal Flush

Starting from the top, or highest value, hand one has the **royal flush**. A royal flush consists of the cards with ranks 10 J Q K A all having the same suit. Evidently, there are precisely 4 such hands.

2. Straight flush not royal

The next specified hand is called the **straight flush not royal**. This hand is beaten by a royal flush but beats every other hand. A straight flush not royal consists of five cards sharing the same suit and having consecutive ranks. Note that such a hand may start with an ace, so that A 1 2 3 4 is the lowest “straight” and 10 J Q K A is the highest “straight.” If the cards are of the same suit, then the highest straight would
be a royal flush. Thus, ruling out the royal flushes, we can index the straight flushes which are not royal (for each fixed suit) by the lowest card A through 9 with 9 10 J Q K as the highest. There are nine

such straights for each suit and four suits for a total of 9 (4) = 36 hands each of which is a straight flush not royal.

Straight flush

Evidently, we can also say at this point that there are 40 hands each of which is a **straight flush**. (This includes the royal flush in each suit.)

3. Four of a kind

The next specified hand is **four of a kind**. Such a hand is beaten by any straight flush but beats any other hand. As the name suggests, the four of a kind hand consists of four cards of the same rank. The fifth card will of necessity have a different rank. These are easy to count: There are 13 ranks possible for the four cards with the same rank (step 1 in constructing a four of a kind hand). Once the rank is chosen, the second step is to determine the last card: There are 52-4 = 48 possible last cards (step 2) for a total of
13 (48) = 624 hands each of which contains four of a kind.

4. Full house

A **full house** consists of three cards of one rank and two cards of a second rank. That is, three of a kind along with a pair. A full house loses to any of the three hands listed above but beats all other hands not listed so far.

5. Flush (not a straight flush)

A **flush** is any hand consisting of five cards having the same suit. The straight flush is an example, but here we exclude the straight flush hands. These **flush not straight flush** hands lose to a full house (and the other hands listed above) but beat the remaining hands listed below.

Flush

The specified hands listed as 1, 2, and 5 together are the hands each of which is a **flush**.

6. Straight (not a straight flush)

A **straight** is a poker hand with 5 cards having consecutive ranks. As mentioned above, the lowest straight consists of cards having ranks A 2 3 4 5, the highest straight consists of cards having ranks 10 J Q K A; if the cards in a straight all have the same suit, then the hand is a straight flush, and such hands are excluded here. Thus a hand is a **straight not flush** if it consists of cards having consecutive ranks but contains cards of at least two suits. Any flush or higher value hand (listed above) beats a straight not flush, but a straight not flush beats all hands listed below.

Straight

The specified hands listed as 1, 2, and 6 together are the hands each of which is a **straight**.

7. Three of a kind

A poker hand is considered to be a **three of a kind** hand if it contains exactly three cards of the same rank. That is, three cards have the same rank, and the remaining (two) cards have a different rank from the first three and also from each other. (If the remaining two cards have the same rank, one has a full house.) A three of a kind hand loses to any of the hands listed above and wins against any of the hands listed below.

This is the first kind of hand in our listing specifically mentioned in the MIT problem under consideration, and we are asked to determine, i.e., count, how many of these hands there are.

8. Two pair

A poker hand is considered to be **two pair** if there are two cards of one rank, two cards of another (different) rank, and a remaining card of a third rank. Naturally, the hands listed above all beat two pair, but two pair still beats the two remaining hands listed below, one of which is the one pair hand mentioned at the outset above.

Two pair hands are also mentioned in the problem, and we are asked to count them.

9. One pair

One pair hands are discussed above. It should be noted that these hands are distinct from the hands listed as two pair (8), three of a kind (7), full house (4), and four of a kind (3). One pair loses to the 8 kinds of hands listed above but beats all the hands in the tenth entry in our listing below.

These are mentioned in the problem, and we will determine how many one pair hands are possible.

10. High card (or no pair)

A **high card** hand is any hand not listed above. Such a hand loses to any hand listed above.

The number of one pair hands

As a first step consider constructing one pair of cards (having the same rank). This itself can be considered a two-step construction with steps
(1a) choose a rank and
(1b) choose the two suits (which determine the pair).

Since there are 13 ranks, there are 13 ways to complete step (1a). Once the rank has been chosen there are 4 choose 2 ways to determine the suits involved. That is, there are C(4,2) = 6 ways to complete step (1b). By the multiplication principle, there are 13 (6) = 78 possible pairs.

We can take as a second overall step, choosing the ranks of the other cards (once the pair has been chosen). Recall that the pair is determined by a rank, and if we do not want to get three of a kind, which we do not, then the ranks of the remaining three cards in the hand must be chosen from among the remaining 12 ranks. Also, the rank of each of these cards must be different from the others, lest we get two pair or a full house. Consequently, there are 12 choose 3 or C(12,3) = 12 (11) (10)/6 = 220 possible choices of the remaining three ranks.

Once the pair is chosen and the ranks of the three remaining cards are chosen, we can take as a third (overall) step the determination of the suits of the remaining three cards. There are 4 choices of suit for each card, and each such choice determines a different hand since the ranks of the last three cards are distinct. Thus, there are 4^3 = 64 possible ways to choose the suits of the three last cards and complete the final step determining the hand.

Applying the multiplication principle to these three steps we see there are

(78 pairs) (220 choices of rank for the remaining three cards) (64 choices of suit for the remaining three cards)

= 1098240 one pair hands.

The number of one pair hands may be computed in different ways. Let us consider one of those ways which gives a reminder of the derivation of counting (with) combinations starting with the more primitive method of counting (with) permutations.

We first count the number of one pair poker hands in which the order of cards (or the order in which the cards are dealt) is distinguished. Let us imagine first that the one pair is dealt in the first two cards. Then there are 52 choices for the first card and 3 choices for the second card of matching rank. Thus, there are 52 (3) = 156 “first two card pairs with the order distinguished.” It may be noted that this number is twice the number of possible pairs given above.

If the first two cards determine the pair, then there are 50 cards remaining in the deck, and the third card must be chosen/dealt from among those cards, but the two cards of the same rank must be excluded to avoid three of a kind. Thus, there are 50 - 2 = 48 possibilities.

For the fourth card, there are 49 cards in the deck and 2 + 3 exclusions: 44 possible fourth cards.

Finally, for the fifth card, there are 48 cards in the deck and 2 + 3 + 3 exclusions: 40 possible fifth cards.

The multiplication principle gives a total of 156 (48) (44) (40) = 13178889 possible one pair hands in which the first two cards are the pair and order is distinguished.

It is possible for the pair to also appear as the first and third card or with various other placements in the hand. There are 5 choose 2 such placements. That is, 5 (4)/2 = 10 of them. Thus, there are

131788890   one pair poker hands in which order is distinguished.

If we want one pair poker hands, then we have overcounted. In fact, each distinct one pair hand has been counted the number of ways there are to arrange any hand of 5 cards (containing a pair). That is P(5,5) = 120. Dividing 131788890 by 120 we get 1098240 one pair hands, just as with the first approach.

The number of two pair hands

Finally, we are ready to start the problem.

There are 78 possibilities for one pair. Recall that this can be calculated by choosing the rank of the pair first (13 choices) and then multiplying by the number of ways the suits in that pair C(4,2) = 6 can be chosen. Notice that the 13 choices for the rank can be interpreted at 13 choose 1 or C(13,1). With this in mind the calculation

C(13,1) C(4,2) = 78

may be generalized to count the number of two pairs (or sets of two distinct pairs or four card hands consisting of two pair).

We first choose two distinct ranks for the two pair. This gives 13 choose 2 or C(13,2) = 78 possibilities. Once the ranks of the two pair are chosen, the suits must be chosen. Since the rank of each pair is different from that of the other pair, the choice of suit is independent for each pair, that is to say, there are the same number of choices for suit in each case. That number is 4 choose 2 or C(4,2) = 6. Thus, the number of two pair (four card hands) is

78 (6)^2 = 2808.

Given a choice of one of these 2808 distinct sets of two pair, each possible choice for the remaining card leads to a distinct hand. There are 48 cards left after the four cards appearing in the two pair are excluded. There are four more cards having the same rank as one of the two pair among those 48 cards which must also be excluded to avoid getting a full house. Thus, there are 44 choices for the “last” card in the hand, which perhaps we should call simply the “card which is not in a pair” since order does not matter. The number of distinct two pair hands is

2808 (44) = 123552.

Exercise: Counting the number of two pair hands a different way, we can say there are 78 possibilities for one pair. A first step in constructing a two pair hand might be determining a first pair. The next pair must have a different rank, so once the first pair is chosen there are 12 possible ranks for the second pair. There are C(4,2) = 6 possible ways to choose the suits for the cards in the second pair. Therefore, there are

12 (6) = 72

possible ways to determine the second pair (and complete the second step). A third step to complete the hand is choosing the final card. Four cards have been removed to determine the third pair leaving 48 cards “in the deck,” and another four cards with matching ranks must be excluded. This leaves 44 possibilities for the fifth card (and the final step). By the multiplication principle there are

78 (72) 44 = 247104 two pair poker hands.

What is wrong with this reasoning?

The number of three of a kind hands

We now generalize our counting of one pair hands to determine the number of three of a kind hands. The three of a kind or triple has, first of all, a rank chosen from among the 13 ranks. For each choice of rank there are 4 choose 3 or C(4,3) = 4 choices of the suits appearing the the triple. Thus, there are

13 (4) = 52 triples.

Once the triple is chosen, there are 49 cards in the deck and two more cards to choose. One of the 49 cards is excluded (the one with the same rank as the triple) to avoid getting four of a kind. From among the 48 remaining cards two must be chosen, and those two must have different rank. Thus, we can take as a second step (after the triple has been chosen) the determination of the ranks of the two remaining cards. There are 12 choose 2 or C(12,2) = 66 possibilities. Once the ranks of these to cards are chosen, the suits for them can be chosen independently giving 4^2 = 16 possibilities. Applying the multiplication principle we get

52 (66) (16) = 54912   three of a kind poker hands.

Comparison of two pair and three of a kind poker hands

There are 123552 two pair hands and 54912 three of a kind hands or exactly "two and a quarter times as many of the former as there are of the latter." In this way, it is natural to say the two pair hands are at least twice as "common" as three of a kind hands. If you look at the set of all two pair hands and the set of all three of a kind hands among the set of all poker hands, you will see more than twice as many two pair hands.

Relative numbers of poker hands

We have now a partially filled in table of the numbers of some different kinds of poker hands:

1. royal flush	4
2. straight flush not royal	36
straight flush	40
3. four of a kind	624
4. full house	
5. flush not a straight flush	
flush	
6. straight not a straight flush	
straight	
7. three of a kind	54912
8. two pair	123552
9. one pair	1098240
10. high card	
total number of all hands	2598960

Exercise: Fill in the rest of the table.

Probabilities

The problem does not actually suggest counting the numbers of hands but rather asks for the "probability" of a two pair hand and a three of a kind hand. We are then asked which is "more likely."

I am going to give rather unorthodox answers to these questions. We can use the word "probability" to attach a number to a particular kind of hand. We already encounter a difficulty here because the word “probability” suggests that the number gives some indication concerning whether the appearance of such a hand (in some particular actual situation) is “probable.” The problem is that, as far as I know, there is no such number.

The particular number Orloff and Bloom have in mind (for the “probability”) is the ratio of all the hands of a specified sort to all the possible hands. For two pair this number is

123552/2598960 ≐ 0.047539.

Here I am using the symbol “≐” to indicate that a decimal approximation has been used.

The ratio of all three of a kind hands to the 2598960 possible hands is

54912/2598960 ≐ 0.0211285.

Since the latter number is smaller, Orloff and Bloom expect me (the reader of their text and student of probability) to conclude a three of a kind hand is “less likely” than a two pair hand. I think this is an incorrect conclusion.

The basic question would be, I think, “under what circumstances is two pair more likely than three of a kind?” Perhaps surprisingly, my answer is: “There are no such actual circumstances.”

One possible real circumstance that might be contemplated is that in which the cards have been dealt in a game of poker (in which I am a player), and I am interested in the particular hand I have been dealt. I look at the hand, and I see three of a kind. In this circumstance, two pair is much less likely than three of a kind: I have been dealt a three of a kind hand.

The natural objection is that this is not the circumstance one should consider, but rather one should consider the circumstance in which a particular hand of poker has yet to be dealt (in a poker game in which I am a player). In this case, it is argued that “it is more likely I will be dealt two pair rather than a three of a kind.” Again, I disagree. I think a more reasonable way to interpret the situation is that the particular deck of cards being used has experienced a sequence of well-defined manipulations (none of which can be reasonably considered in any way magical or “random”) resulting in a particular arrangement of the cards when they are dealt and consequently a particular hand I will be dealt when the cards are distributed. It may indeed be said that I do not know the hand I will be dealt. But knowing the numbers 0.047539 and 0.0211285 does not help me answer that question nor the question of which is more likely. The fact of the matter is simply that I do not know the manipulations to which the deck has been subjected in detail. Were I to know those details adequately, then I could assert that the “likelihood” I will be dealt two pair is answerable in one of two ways: Either I will be dealt two pair or not. The only reasonable numbers to attach to the “likelihood” I will be dealt two pair are 1 (I will be dealt two pair) and 0 (I will not be dealt two pair). I simply do not know which number associated with the likelihood is the correct one.

At this point, I suspect I may be accused of misunderstanding (or misrepresenting) the appropriate circumstance for application of the numbers 0.047539 and 0.0211285. That may be the case, but for the moment perhaps the best I can do is anticipate the objection and give some explanation or answer concerning my (counter) objection. The **real** circumstance I should be considering (I suspect it may be suggested) is that in which I am playing a **very large number** of games (or hands) of poker. As I play those games, I can count the number of times I am dealt two pair. Specifically, if I have played n games, then I can keep track of the number $\mathcal{N}$(n). Then if I look at the ratio

$\mathcal{N}$(n)/n

I will find **this quantity tends to** 0.047539 **as** n **tends to infinity**. Thus, for example, if I intended to devise a strategy for betting against another player, it may be helpful for me to incorporate the number 0.047539 in regard to the appearance of two pair hands (as well as other similarly calculated "probabilities.") This then would be the real meaning of the number 0.047539 called (or somewhat improperly called) the "probability" of being dealt two pair as a hand.

This is a somewhat interesting assertion. I will admit that this description is extremely suggestive and rather compelling. I might even be tempted to admit that I am inclined to think it is correct or at least that the number 0.047539 is a reasonable number to associate with this kind of hypothetical situation. I still see some problems however. These problems start with the assertion that this is something "I will find." The fact is that I will not find this, and I will not find it for various reasons. Here are two of the most obvious:

1. I only have so much time in my life, so it is impossible for me to play infinitely many games of poker. In short, the circumstance that I play infinitely many games of poker is not a real circumstance. There is no way to verify (that is for me to "find") the basic assertion.
2. Even were I to invest a large amount---say all---of my time playing poker so that the number n might be considered somehow "large," my objection applying to the previous circumstances seems to still apply: The value of the ratio $\mathcal{N}$(n)/n is not the result of anything magical or random. It is the result of a particular sequence of manipulations of the cards, or various decks of cards. Those manipulations may be complicated and I may not know them in detail, but the fact that I do not know them does not justify the expectation of any particular likelihood for the ratio $\mathcal{N}$(n)/n any more than these circumstances (a complicated process plus my ignorance of the details of that process) might justify any expectation with respect to the outcome of a single hand. The ratio $\mathcal{N}$(n)/n for each n is going to be what it is going to be, and I have no way to predict that ratio.

Here are two that are perhaps less obvious:

3. I do not want to invest a large amount of my time playing poker. That would be a mistake.
4. Even if I could make a lot of money playing poker (due to my superior understanding of how these ratios, both the hypothetical ones $\mathcal{N}(n)/n$ for large n and the calculated ones like 0.047539, are asserted to be, beieved to be, and possibly are related) this would still be a mistake for various reasons, e.g., I would be taking someone else's money just because he is not as smart as I am---according to my evaluation, I would be immorally taking advantage of that person.
5. The **opportunity cost** of what I might be doing and accomplishing by participating in constructive and moral activities rather than playing games and trying to trick other people is, by my estimation, rather significant.
5. I am unaware of any circumstance in which there is a truly desirable outcome which has been obtained as the result of actions based on probabilistic or statistical analysis. Of course, this requires a value judgement concerning which outcomes are desirable and which are not. This is my value judgement.

Hands with at least one pair

For comparison and to exercise our counting skills once more, let us consider the possibility of getting "at least one pair," meaning we count also "three of a kind" hands, "four of a kind" hands, "two pair" hands, and "full house" hands. In this case, let us count the number of hands in which there is no pair.

A hand with no pair (by which in this case we mean something different from a high card hand) must involve 5 distinct ranks. Thus, such a hand can be constructed in two steps by first determining those 5 ranks. There are 13 choose 5 or C(13,5) = 1287 possibilities for the five ranks. Once those are chosen, there are 4 independent possibilities for the suit of each card with each choice resulting in a distinct hand. Thus, there are

1287 (4)^5 = 1317888   hands with no pair

and

2598960 - 1317888 = 1281072   hands with at least one pair.

To check this calculation, we should be able to add up all the numbers of hands with one pair, two pair, three of a kind, a full house, and four of a kind. In fact,

```
In[ ]:= 1 098 240 + 123 552 + 54 912 + 3744 + 624
Out[ ]= 1 281 072
```

References

Orloff and Bloom, Class 1 Reading(s), Introduction to Probability and Statistics, MATH 18.05 MIT OpenCourseWare, Spring 2014

The Wikipedia page on "Poker Probability"