Excerpt: Appendix B Functions

from Alexander Farmer's book *The Imposition of Dystopia: Probability and Statistics*

J. McCuan

September 1, 2023

Note: This is copied directly with a few minor changes of notation.

Here is a working definition of the mathematical term **function** many have found illuminating:

> Given sets X and Y, a **function** is a rule or correspondence which assigns to each $x \in X$ a unique $y \in Y$.

This can be made, mathematically and set theoretically precise, in the following, possibly less illuminating, way

Definition 1 Given sets X and Y, a **function** is any subset R of the Cartesian product $X \times Y$ having the following properties:

(i) For each $x \in X$, there is some $y \in Y$ with $(x, y) \in R$, and

(ii) If $(x, y) \in R$ and $(x, z) \in R$, then $y = z$.

The set X in this definition is called the **domain** of the function. The set Y in this definition is called the **codomain** of the function. The set

$$\{y \in Y : \text{ there exists some } x \in X \text{ with } (x, y) \in R\}$$

is called the **range** of the function.

This is called the **relational definition** of a function in which a relation[1] $R \subset X \times Y$ plays a central roll. It has the nice quality that the function itself is given a name R. Essentially all aspects of the relational definition carry over to the more informal definition given above in which the function is a (nameless) rule. The "rule" definition has some nice qualities too. We can give the rule a name, and that name is usually f. Then we can introduce the nice notation $y = f(x)$ to mean the ordered pair (x, y) is in the relation. We also gather all the important notational entities together in the suggestive formulation

$$f : X \to Y$$

[1]A **relation** is simply a subset of the Cartesian product $X \times Y$.

which means f is a function (of the "rule" sort) with domain X and codomain Y. We also say (suggestively) "f is a function from X to Y."
Note: Whenever we write $f(x)$, it is implicit (and implied) that x is an element of the domain of f.

Injective, surjective, and bijective

A function $f : X \to Y$ is **injective** or **one-to-one** if $f(x) = f(\xi)$, then $\xi = x$.

Exercise 1 Condition **(ii)** in the relational definition of a function given above is sometimes called the **vertical line test**.

(a) Rephrase the vertical line test using the "rule" notation $f(x)$.

(b) Rephrase the injectivity condition using the relational notation $(x, y) \in R$ for a function.

(c) Explain injectivity in terms of a **horizontal line test**. Hint: For both the vertical line test and the horizontal line test, the set/relation

$$\{(x, f(x)) : x \in X\}$$

may be useful to consider.

Exercise 2 Give an example of a function for which the range and the codomain are not the same sets.

Given a function $f : X \to Y$, the set

$$\mathcal{G} = \{(x, f(x)) : x \in X\} \subset X \times Y$$

is called the graph of the function f. (Where have you seen this set appearing under a different name?)

A function $f : X \to Y$ is **surjective** or **onto** if for each $y \in Y$, there exists some $x \in X$ with $y = f(x)$.

Exercise 3 Explain surjectivity in terms of a **horizontal line test**. Note: There is a horizontal line test for injectivity and a horizontal line test for surjectivity.

Exercise 4 There is a vertical line test for condtion **(ii)** in the relational definition of a function. What else is there a vertical line test for?

A function $f : X \to Y$ is **bijective** or **one-to-one and onto** if f is both injective and surjective. Such a function is called a **one-to-one correspondence**. Such a function has an **inverse**.

Definition 2 Given a function $f : X \to Y$ and a function $g : Y \to X$, the function g is said to be an **inverse** for the function f if the **compositions**

$$g \circ f : X \to X \text{ by } g \circ f(x) = g(f(x)), \text{ and}$$
$$f \circ g : Y \to Y \text{ by } f \circ g(y) = f(g(y))$$

satisfy the conditions

$$g \circ f(x) = x \text{ for every } x \in X, \text{ and}$$
$$f \circ g(y) = y \text{ for every } y \in Y.$$

Exercise 5 Prove that if a function $f : X \to Y$ has an inverse $g : Y \to X$, then the inverse g is unique. Hint(s): Consider another function $h : Y \to X$ which is an inverse of f, and show $h = g$. Two functions are equal if all their values are equal or equivalently if they are the same relation/have the same graph.

Exercise 6 Prove a function $f : X \to Y$ has an inverse if and only if f is bijective.

The function from X to X corresponding to, i.e., given by, the diagonal relation

$$\{(x, x) : x \in X\}$$

has a special name and notation. This is the **identity function** and is denoted by id or id_X. Notice that the conditions in the definition of the inverse g of a function f may be rephrased as

$$g \circ f = \text{id}_X \qquad \text{and} \qquad f \circ g = \text{id}_Y.$$