Excerpt: Appendix A Sets and Cardinality

from Alexander Farmer's book *The Imposition of Dystopia: Probability and Statistics*

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Note: This is copied directly with a few minor changes of notation.

Most texts on probability and statistics include some brief discussion of sets. I will include some of the standard material usually mentioned with a little more detail.

There is a mathematical subject (or at least semi-mathematical or sub-mathematical subject) called **set theory**. A possible first axiom of set theory is the **axiom of existence**:

$$\text{There exists a set.} \tag{1}$$

Mathematicians were preaching mathematical dogma long before the formulation of the axiom of existence. Apparently none of those mathematicians felt it necessary to state the axiom (1). Many of them believed in sets. If you believe in sets too, then the properties, terminology, and notation associated with them may provide you and another believer with a convenient means of communication. While it may be cumbersome to start with (1), a review of some of the things many mathematicians take for granted may be helpful.

1 Belonging and exclusion

The basic property of a **set** A is the following: One can determine if any "object" under discussion (which in the mathematical context essentially always means another set) "belongs" to A. If an "object" x belongs to the set A, we say x is **an element** of A and write

$$x \in A. \tag{2}$$

Otherwise, we should be able to conclude that x is "excluded" from A in which case we write

$$x \notin A \tag{3}$$

and say x is **not an element** of A. In situations where it is found that both (2) and (3) hold for sets x and A, a mathematician traditionally displays a strange facial expression and declares the occurrence of a **contradiction**. Such occurrences are not to be taken seriously (mathematically) for extended periods of time.

Definition 1 A set A is a **subset** of a set B and we write $A \subset B$ if

> $x \in A$ implies $x \in B$ (for every $x \in A$).

Definition 2 Two sets A and B are **equal** if $A \subset B$ and $B \subset A$, i.e., if A and B have precisely the same elements.

2 Sentences and specification

Another submathematical subject we may wish to take for granted is called **logic**. This subject involves "statements" or "sentences" σ composed of various **qualifiers**,[1] **conjunctions**,[2] and **implications**.[3] When a sentence σ depends on a set x, we may write $\sigma = \sigma(x)$ and invoke the **axiom of specification**:

> There exists a subset of A containing precisely the elements in A for which $\sigma(x)$ holds (or is "true").

In this case, we can denote the newly constructed set by

$$\{x \in A : \sigma(x)\}$$

which is read "the set of all x in A for which $\sigma(x)$ holds." Incidentally, the symbol σ is the Greek letter (lower case) "sigma" and may be considered a kind of equivalent of the English/Roman letter s.

Note: At some point it may be desirable to try to communicate using more or less formal "proofs" (or strings of reasoning). Using the phrase "by the same logic" while attempting such a feat is usually frowned upon. From the point of view of mathematical communication, it is assumed (or imagined) the participants are all using one and the same "logic." The phrase "by the same reasoning," or even better "using similar reasoning," is much preferred. In the latter it may be imagined logic is essentially universal and reasoning is a particular arrangement of logical conclusions. Different people are quite capable of producing (and to a certain extent capable of communicating) distinct arrangements of logical statements. When different people are using possibly different "logic," then communication is probably hopeless.

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[1] For example: (for) some, (for) all, not.

[2] For example, "or" and/or "and."

[3] These have the form "if ____________, then ____________" and/or "____________ implies ____________."

3 Unions, intersections, complements

In set theory there are also unions: If A and B are subsets of a set S, then

$$A \cup B = \{x \in S : x \in A \text{ or } x \in B\}. \tag{4}$$

Note: Technically, an **axiom of unions** is "needed" to assert the existence of the set S in the specification (4). More generally, a **superset** S is usually required in any specification. Some indication of why this is the case is suggested by Exercise 2 below. We may informally write something like $\{x : \sigma(x)\}$, but we need to be careful to make sure there is some superset to which all the specified elements belong.

More generally, if $\mathcal{C}$ is a family of sets, i.e., a set of sets, each of which is a subset of a given set S, then there exists a unique set

$$\bigcup_{A \in \mathcal{C}} A = \{x \in S : x \in A \text{ for some } A \in \mathcal{C}\}.$$

This set is called the **union** of $\mathcal{C}$ or the **union of all** $A \in \mathcal{C}$. Again, some version of the axiom of unions is needed in general to assert the existence of the superset. The union of all $A \in \mathcal{C}$ is occasionally denoted by $\cup\mathcal{C}$.

Similarly, if $\mathcal{C}$ is nonempty, we can specify in $S = \cup_{A \in \mathcal{C}} A$ a unique set

$$\bigcap_{A \in \mathcal{C}} A = \{x \in S : x \in A \text{ for every } A \in \mathcal{C}\}.$$

This set is called the **intersection** of the sets $A \in \mathcal{C}$.
Note: The intersection may be **empty**, i.e., may contain no elements. This is a special set called the **empty set** and denoted variously by some symbol like ϕ or $\emptyset$.

Exercise 1 Use the axiom of existence and the axiom of specification to show there exists a (unique) set ϕ with no elements.

Exercise 2 Consider

$$B = \{A : A \text{ is a set and } A \notin A\}.$$

(a) What happens if $B \in B$?

(b) What happens if $B \notin B$?

(c) What do you conclude from your answers to parts **(a)** and **(b)** above?

Exercise 3 While we mentioned an axiom of unions, why (do you think) I did not mention an axiom of intersections?

4 Cartesian products

The **Cartesian product** of two sets A and B, denoted $A \times B$ is the set of all **ordered pairs** (a, b) of elements $a \in A$ and $b \in B$. In terms of (the axiom of) set specification

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}. \tag{5}$$

In the special case where $B = A$, the Cartesian product of A with itself is denoted A^2. This is how the beloved notation $\mathbb{R}^2$ arises for the beloved Cartesian plane

$$\{(x, y) : x, y \in \mathbb{R}\}$$

in the beloved subjects of analytic geometry and calculus.

For those who might be curious, an "axiom of products" is not needed. What is needed, however, is some kind of expression making it clear that the ordered pair (a, b) is a set—as the only kinds of objects really under consideration in set theory are sets.[4] One possibility is $(a, b) = \{ \{a\}, \{a, b\} \}$. You may puzzle out for yourself why this might be a good choice and what the superset in (5) might be.

As you might guess at this point there is a generalization for products along the following lines: If $\mathcal{C}$ is an **ordered collection of sets**, then

$$\prod_{A \in \mathcal{C}} A = \{(x_A) : x_A \in A\}$$

is the Cartesian product of the sets in $\mathcal{C}$. This construction is most often used when $\mathcal{C}$ is a **finite** set, i.e., there is some **natural number** n for which $\mathcal{C} = \{A_1, A_2, \ldots, A_n\}$ and

$$\prod_{j=1}^{n} A_j = \{(x_1, x_2, \ldots, x_n) : x_j \in A_j \text{ for } j = 1, 2, \ldots, n\}.$$

Technically, the natural numbers $\mathbb{N}$ are usually defined after the Cartesian product of only "two" sets is defined separately. Only ordered pairs and ordered pairs of ordered pairs are needed to construct natural numbers which look something like this:

$$1 = \{\phi\},\ 2 = \{\phi, \{\phi\}\} = \{\phi, 1\},\ 3 = \{\phi, 1, 2\},\ \ldots.$$

Then one can proceed to define the "larger" products using natural numbers.

Occasionally, we may also consider a **countable product** of sets which is a Cartesian product

$$\prod_{j=1}^{\infty} A_j = \{(x_1, x_2, x_3, \ldots) : x_j \in A_j \text{ for } j = 1, 2, 3, \ldots\}$$

corresponding to a **sequence** $A_1, A_2, A_3, \ldots$ of sets.

[4] In this regard, a quote from page 1 of the book **Naive Set Theory** by Paul Halmos may be of interest: "By way of examples we might occasionally speak of sets of cabbages and kings, and the like, but such usage is always to be considered as an illuminating parable only, and not part of the theory..." In reference to Exercise 2 above, one may also wish to consult Halmos' "more spectacular" statement at the top of page 7 in the same book.

5 Complements

Given a fixed superset S, the **complement** of a set $A \subset S$ or the **complement of A with respect to S** is

$$A^c = \{x \in S : x \notin A\}.$$

Notice that to use the notation A^c for the complement, the superset S must be known and/or understood. A more general construction is the **relative complement** of A with respect to a set B given by

$$B \backslash A = \{x \in B : x \notin A\}.$$

This (construction) does not require, and the notation $B \backslash A$ does not imply, the condition $A \subset B$. Some texts use the notation A' for the complement A^c and the notation $B - A$ for the relative complement $B \backslash A$.

6 De Morgan's laws

There are various "operations" one can use on/apply to pairs and other collections of sets. Two of the most interesting are De Morgan's laws:

$$(A \cup B)^c = A^c \cap B^c, \text{ and} \tag{6}$$
$$(A \cap B)^c = A^c \cup B^c. \tag{7}$$

In order to verify or prove De Morgan's laws, we should use the definition of set equality. For example, to show De Morgan's first law (6) we should show

$$\begin{aligned} (A \cup B)^c &\subset A^c \cap B^c, \text{ and} \\ A^c \cap B^c &\subset (A \cup B)^c. \end{aligned}$$

Thus, we take an element $x \in (A \cup B)^c$. We then know $x \notin A \cup B$. This implies $x \notin A$: If $x \in A$, then $x \in A \cup B$, and we have a contradiction. Similarly, $x \notin B$. All together, we have $x \in A^c$ and $x \in B^c$, and this means $x \in A^c \cap B^c$ which shows

$$(A \cup B)^c \subset A^c \cap B^c.$$

For the reverse inclusion, we take an element $x \in A^c \cap B^c$. We then know $x \notin A$ and $x \notin B$. Were we to assume

$$x \in A \cup B = \{a \in S : \ a \in A \text{ or } a \in B\},$$

then we have an immediate contradiction, which means $x \notin A \cup B$ so that $x \in (A \cup B)^c$, and

$$A^c \cap B^c \subset (A \cup B)^c$$

as we needed to show. When we complete a proof like this, we can put a special symbol to wake up all those who have fallen asleep. One possibility is the following: $\square$

Exercise 4 Prove De Morgan's second law.

A note on the use of sets in studying probability: The text *Probability and Statistical Inference* by Hogg, Tanis, and Zimmerman which is an introductory text on probability and statistics contains the hopelessly absurd suggestion that "in studying probability the words *set* and *event* are interchangeable" (italics in the original). Hopefully, it is somewhat clear from the presentation above that **sets** are mathematical constructions, i.e., imaginary or linguistic pictures, based perhaps on some axiomatic set theory. The notion may be extended informally, or as a kind of parable, to sets of actual physical objects, e.g., sets of cabbages and kings. **Events** are something quite different. Certainly, one might **informally** imagine a collection of events, things that actually happen in the real physical world, as a set and consequently a subset of such a set of events as somehow an "event" itself—and thus a set. Even embracing such confusing and useless informality, there are other (informal) sets, like a set of cards, that are not events.

A much more natural approach to "studying probability," if such a thing is actually to be considered, is to say that **events may be modeled by sets**. That is to say, one can impose, or propose, some kind of correspondence and comparison between some events and certain sets. The sets may be imagined to provide some kind of mathematical model for events, but the words "set" and "event" are (clearly) not (or at least clearly should not be made) interchangeable.

7 Cardinality

Roughly speaking the **cardinality** of a set is the number of elements in the set. This works quite well for sets with **finite cardinality**, that is sets which can be put in one-to-one correspondence with a natural number (in the set theoretic sense) or a particular specified set concerning which there is general agreement about the number of elements contained in it. Recall that the **natural numbers**

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

are supposed to be a collection of sets having this property with $1 = \{\phi\}$ having one element, $2 = \{\phi, 1\}$ having two elements, $3 = \{\phi, 1, 2\}$, and so on. To make this more precise, we need a solid definition of a **one-to-one correspondence** which can be found in the next section/appendix.

Assuming you know about functions and bijections, e.g., from the next section/appendix, we say a set A has **cardinality** 1 if there is a one-to-one correspondence $f : A \to 1$. In this case we write

$$\#A = 1.$$

Similarly, a set A has cardinality $n \in \mathbb{N}$ and we write $\#A = n$ if there is a one-to-one correspondence $f : A \to n$.

Definition 3 A set A is said to have **finite cardinality** if $\#A = n \in \mathbb{N}$, that is, there exists some natural number $n \in \mathbb{N}$ and a bijection $f : A \to n$.

It may not quite make sense to say sets which do not have finite cardinality "have the same number of elements," but our basic definition of cardinality can be extended to "larger" sets. Specifically, two sets A and B are said to have the **same cardinality** and we write $\#A = \#B$ if there exists a bijection $f : A \to B$. In particular, the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ does not have finite cardinality, but $\mathbb{N}$ serves as a standard set with respect to cardinality: A set A is said to be **countably infinite** or just **countable** and we write $\#A = \aleph_0$ if there is a bijection $f : A \to \mathbb{N}$.

Much more can be said on this subject. For the time being, let it suffice to observe that to count the number of elements in a set of finite cardinality is to determine the cardinality of that set. We consider some techniques for counting the number of elements in various sets of finite cardinality in Appendix C.

8 Sets of numbers

The construction of various sets of numbers may be considered part of set theory. These sets are probably familiar to you and the details are (or turn out to be) relatively complicated. For these reasons, I will not discuss the details, but simply give a list of the sets with their names and standard symbols for reference.

natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$

natural numbers with zero $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$

integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

rational numbers

$$\mathbb{Q} = \left\{ \frac{m}{n} : \ m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \right\}$$

real numbers $\mathbb{R}$

Note: This set of numbers while very familiar is quite complicated to define precisely and simply. You may know $\mathbb{R}$ as "the real line" or "the interval from $-\infty$ to ∞." Here are some properties which should agree with what you know about $\mathbb{R}$:

(i) $\mathbb{Q} \subset \mathbb{R}$. In fact, $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q}$.

(ii) (ordering) Given $a, b \in \mathbb{R}$ exactly one of the following holds: $a < b$, $a = b$, or $a > b$. The sets $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{Q}$ all have an order like this as well.

(iii) (Archimedean property) For each $a \in \mathbb{R}$, there is some $b \in \mathbb{R}$ with $b > a$. The sets $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{Q}$ all satisfy the Archimedean property as well.

(iv) (Dedekind completeness) If $B \subset \mathbb{R}$ is nonempty and bounded above, i.e., there is some $M \in \mathbb{R}$ for which $b \leq M$ for every $b \in B$, then there exists a **least upper bound** $m \in \mathbb{R}$ for the set B, i.e., $b \leq m$ for every $b \in B$ and if $b \leq \alpha$ for every $b \in B$, then $m \leq \alpha$.

Exercise 5 Which of the sets $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{Q}$ are Dedekind complete?

extended real numbers

$$[-\infty, \infty] = \{x \in \mathbb{R}\} \cup \{\pm\infty\} \text{ and } (-\infty, \infty] = \{x \in \mathbb{R}\} \cup \{\infty\}.$$

You may contemplate which properties of $\mathbb{R}$ extend to $[-\infty, \infty]$ and $(-\infty, \infty]$. Notably, **addition** extends to $(-\infty, \infty]$ but not to $[-\infty, \infty]$, though we didn't address the algebraic properties of any of these sets of numbers, like the existence of an operation of addition.

complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Note that $\mathbb{R} \subset \mathbb{C}$, and $\mathbb{C}$ is not ordered. In particular, $\mathbb{C}$ cannot satisfy the Archimedean property, nor can $\mathbb{C}$ be considered Dedekind complete. The set $\mathbb{C}$ does inherit some related properties from $\mathbb{R}$. In particular, $\mathbb{C}$ is **metrically complete**. I will not discuss what this means, but you can think about it (or look it up).

All the sets of numbers we will use are subsets of $\mathbb{C}$ and most are subsets of $\mathbb{R}$. I will mention two more sets of numbers which may be of interest. The **algebraic numbers** are numbers $\zeta \in \mathbb{C}$ such that there exists a **polynomial**

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $n \in \mathbb{N}\backslash\{0\}$ with $p(\zeta) = 0$. The **transcendental numbers** are the complement of the algebraic numbers (with respect to $\mathbb{C}$).