Assignment 3: Counting and Probability
Due October 3, 2023

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Problem 1 Here is a story:

Each individual in group A lives on one acre of land and calls it his own. Furthermore, each individual in group A depletes resources on his acre either by cutting down the forest or by mining. An acre on which resources have been depleted in either manner is called a "depleted acre."

Elsewhere, each individual in group B lives on one acre of land and calls it his own, and there are the same number of individuals in group A and in group B.

(a) 200 individuals in Group A cut down the forest and 50 individuals in group A mine. Out of these individuals in Group A, 25 deplete resources in both ways, cutting down the forest and mining on an acre. How many individuals are there in group A (and how many acres are rendered "depleted acres")?

(b) A human livestock manager observes the outcome of the activity of the individuals in group A and is not happy with the number of "depleted acres" resulting from this activity. He assumes the outcome for the land on which the individuals from group B will be the same without intervention, and he devises a system of permitting to reduce the number of "depleted acres." He issues 200 permits for cutting down forest and 50 permits for mining. Among the permits issued to people in Group B, $25 + n$ of the permits are for both mining and cutting down forest. (The number n is positive, and not every individual in group B is issued a permit.) How many acres does the human livestock manager expect will become "depleted acres" due to the activity of individuals in group B?

(c) Given the specifics above concerning group B, what is the greatest number of acres the human livestock manager might expect to remain outside the "depleted acres" category?

Problems 2-5 below are about the three six sided dice described in Problem 6 of Assignment 2:

(i) One die is red and has five sides marked "three" and one side marked "six."

(ii) The second die is green and has one side marked "one" and five sides marked "four."

(iii) The third die is blue with three sides marked "two" and three sides marked "five."

Problem 2 (intransitive dice; Assignment 2 Problem 6)

(a) How many ordered outcomes are there for rolling the red die first and the green die second? Hint: Introduce the base space $S = \{\text{one}, \text{two}, \text{three}, \text{four}, \text{five}, \text{six}\}$ to model the outcomes for each die (in conjunction with appropriate specification functions $r : S \to \{3, 6\}$ and $g : S \to \{1, 4\}$).

(b) How many of the outcomes considered in part **(a)** are outcomes for which the number showing on the red die is greater than the number showing on the green die?

Problem 3 (intransitive dice; Assignment 2 Problem 6)

(a) How many ordered outcomes are there for rolling the red die first and the blue die second?

(b) How many of the outcomes considered in part **(a)** are outcomes for which the number showing on the red die is greater than the number showing on the blue die?

Problem 4 (intransitive dice; Assignment 2 Problem 6)

(a) How many ordered outcomes are there for rolling the green die first and the blue die second?

(b) How many of the outcomes considered in part **(a)** are outcomes for which the number showing on the green die is greater than the number showing on the blue die?

Problem 5 (product measure) Instead of modeling the competition between the red die and the green die as suggested by the hint in Problem 2, consider the measure spaces

$$S_r = \{3, 6\} \qquad \text{and} \qquad S_g = \{1, 4\}.$$

(a) Define probability measures $\pi_r : \wp(S_r) \to [0, 1]$ and $\pi_g : \wp(S_g) \to [0, 1]$ by giving the values on the corresponding singleton sets compatible with the induced measures of Problem 2.

(b) Write down the values of the product measure $\pi : \wp(S_r \times S_g) \to [0, 1]$ on the singleton sets Ω_{rg} of $S_{rg} = S_r \times S_g$.

(c) How many singletons in Ω_{rg} correspond to outcomes in which the red die shows a greater value than the green die?

(d) Calculate $\pi(\{(\omega_r, \omega_g) \in S_{rg} \ : \ \omega_r > \omega_g\})$.

(e) Calcualte the integral of $x : S_{rg} \to \mathbb{R}$ by $x(\omega_r, \omega_g) = \omega_r - \omega_g$ with respect to the product measure π.

Problem 6 (standard dice; section 3.6 in my notes)

(o) Consider rolling n one sided (!) dice. Count the possible outcomes. The sequence you get $\{z_n\}_{n=1}^{\infty}$, depending on $n = 1, 2, 3, \ldots$, is the sequence of **zero dimensional triangular numbers**.

(a) Consider rolling n two sided dice (i.e., flipping n coins) with the number of "ones," "heads," or "1's" as the outcome.

(i) Count the number of possible outcomes. The sequence $\{a_n\}_{n=1}^{\infty}$ you get is the sequence of **one dimensional triangular numbers**.

(ii) Note that $a_n = 1 + z_1 + z_2 + \cdots + z_n$.

(iii) Find a formula for a_n as a combination.

(b) Consider rolling n three sided dice with a the outcome given by ν_j faces showing j for $j = 1, 2, 3$ and $\sum \nu_j = n$.

(i) Count the number of possible outcomes as follows:

One possibility is n faces show 3	:	1
$\left\{\begin{array}{l}\text{If } n-1 \text{ faces show 3,} \\ \text{the remaining face has the outcomes of a coin flip}\end{array}\right.$	:	a_1
$\left\{\begin{array}{l}\text{If } n-2 \text{ faces show 3,} \\ \text{the remaining faces have outcomes of flipping two coins}\end{array}\right.$	:	a_2
$\vdots$		$\vdots$
$\left\{\begin{array}{l}\text{If no face shows 3,} \\ \text{the remaining outcomes are the same as flipping } n \text{ coins}\end{array}\right.$	:	a_n

The sequence $\{b_n\}_{n=1}^{\infty}$ you get is the sequence of **two dimensional triangular numbers**. (You may want to write the first few.)

(ii) Interpret b_n as, i.e., in terms of, an arrangement of bowling pins.

(iii) Find a formula for b_n as a combination.

(c) Define and interpret the **three dimensional triangular numbers** or **tetrahedral numbers** as, i.e., in terms of, a stack of cannon balls.

(d) Find a formula for the number of outcomes of rolling n indistinguishable m sided dice.

Problem 7 (Problem 6 above)

(a) Where do the $m-1$ dimensional triangular numbers appear in Pascal's triangle?

(b) Count the number of ways to sort n indistinguishable objects into m boxes.

(c) Interpret your answer to part **(b)** in terms of standard dice.

(d) Interpret your answer to part **(b)** in terms of the set

$$\left\{(\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{N}_0^4 \ : \ \sum_{j=1}^{4} \nu_j = 10\right\}.$$

Problem 8 (hands of cards) If you have a deck of n (distinct) cards and k is a natural number with $2k < n$, how many distinct hands containing k cards is it possible to deal using this deck of cards?

Problem 9 (poker hands) Determine the following

(a) The number of "full house" hands.

(b) The probability of being dealt a "full house."

(c) The number of "straight" hands.

Problem 10 (probability urns) Assume you have an urn containing nine balls. Three are red, three are green, and three are blue. Take as an event extracting three balls from the urn without replacement, and as an outcome the three (not necessarily distinct) colors of the balls that have been extracted without regard to order and assuming balls of the same color are indistinguishable.

(a) Give a mathematical set S that models the possible outcomes.

(b) Define a probability measure π on your set S making it a measure space that determines the probability of each possible outcome.

(c) Identify your measure π as constructed from more elementary measures. Hint: Start by assuming the balls are numbered with the outcome of a single extraction of a ball modeled by $S_0 = \{1, 2, 3, \ldots, 9\}$, and build up your measure π from an appropriate probability measure π_0 on S_0.