Assignment 1: Sets, Functions, Measures
Selected Solutions: Problem 2

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September 7, 2023

Problem 1 (From section 1.4.1 of my notes) Let $\mathcal{F}$ denote a family of sets.

(a) Prove De Morgan's law

$$\left(\bigcup_{A \in \mathcal{F}} A \right)^c = \bigcap_{A \in \mathcal{F}} A^c.$$

(b) (counting) If $\#\mathcal{F} = m < \infty$ so that $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ and

$$\#A_j = n_j \qquad \text{for} \qquad j = 1, 2, \ldots, m,$$

then find the cardinality of the Cartesian product

$$\prod_{A \in \mathcal{F}} A.$$

Problem 2 (From section 1.4.2 of my notes) Let $x : R \to S$ be a function.

(a) State precisely what it means for x to be injective.

(b) Show x is injective if and only if there exists a function $y : S \to R$ such that $y \circ x = \text{id}_R$.

(c) State precisely what it means for x to be surjective.

(d) Show x is surjective if and only if there exists a function $y : S \to R$ such that $x \circ y = \text{id}_S$.

Partial Solution:

(a) The function $x : R \to S$ is **injective** if whenever one has $x(a) = x(b)$ for some $a, b \in R$, then $a = b$.

(b) If $x : R \to S$ is injective, and there is some element $a_0 \in R$, then consider $y : S \to R$ by

$$y(c) = \begin{cases} a, \text{ where } x(a) = c, & \text{if } c \in x(R) \\ a_0, & \text{if } c \in S \backslash x(R). \end{cases} \tag{1}$$

Recall that $x(R) = \{x(a) : a \in R\}$. Therefore, if $c \in x(R)$, then there exists some $a \in R$ with $x(a) = R$. Furthermore, S can be written as a disjoint union $S = x(R) \cup [S \backslash x(R)]$, so if we can assign a value $y(c) \in R$ to each element c in one of the sets $x(R)$ and $S \backslash x(R)$ as we have attempted to do in (1) then we will have defined a fucntion $y : S \to R$.

In order to show the function given in (1) is **well-defined**, we should consider the possibility that given $c \in x(R)$ with $c = x(a)$, there is also some $b \in R$ with $x(b) = c$ (for some b for which maybe $b \neq a$). In this case, because the function x is injective and $x(a) = c = x(b)$, we know

$$b = a,$$

so the value given for $y(c)$ in the first case of the definition (1) is unique.

Technically, it might also be the case that the set R is empty, and there is no element $a_0 \in R$. In this situation, the second case of (1) has a problem, and also the entire assertion of part **(b)** of the problem (very likely) has a problem. To be specific, if $R = \phi$, then the function x is what is called the **empty function**. If it happens to be the case that the set S is empty as well, then we are okay: Then we can take $y : \phi \to \phi$ to be the empty function as well. The composition $y \circ x$ is the empty function and the identity on the empty set is the empty function, so the assertion of the problem is trivially, and somewhat vacuously, true. If, however, $S \neq \phi$, then there does not exist any function $y : S \to R = \phi$ (because there is some element $c \in S$, and there is no way to assign this element c to any element of $R = \phi$). In this situation where $R = \phi$ and x is the empty function, however, the following meta-principle comes into play:

> It is difficult to imagine any situation in which it is useful or interesting to consider the empty function.

In view of this meta-principle, I suggest we ignore the case when $x : R = \phi \to S$ is the empty function, though in practice it's a little too late for that.

Returning to the situation in which $R \neq \phi$ and $x : R \to S$ is not the empty function, we have a well-defined function $y : S \to R$ given by (1), and it remains to verify the composition $y \circ x$ is the identity on R. In fact, if $a \in R$, then

$$y \circ x(a) = y(x(a)) = a = \mathrm{id}_R(a).$$

This completes the proof of the assertion that if $x : R \to S$ is injective, then there exists a function $y : S \to R$ for which $y \circ x = \mathrm{id}_R$, i.e., there exists a left inverse (at least in the situation where $R \neq \phi$).

Next we assume the existence of a function $y : S \to R$ with $y \circ x = \mathrm{id}_R$. Take $a, b \in R$ with $x(a) = x(b)$. Applying the function y to the common value $c = x(a) = x(b)$, we see

$$a = \mathrm{id}_R(a) = y \circ x(a) = y \circ x(b) = \mathrm{id}_R(b) = b.$$

This means x is injective. $\square$

Problem 3 (From section 1.4.3 of my notes) Given a (baby) measure μ on a set $S = \{\omega_1, \omega_2, \ldots, \omega_n\}$, explain why the restriction

$$\mu\big|_{\Omega} \qquad \text{to} \qquad \Omega = \{\ \{\omega_1\}, \{\omega_2\}, \ldots, \{\omega_n\}\ \}$$

is **not** a measure.

Problem 4 (Exercise 1.4.3 from my notes) Let $S = \{\omega_1, \omega_2, \ldots, \omega_n\}$ be a set with n elements.

(a) Define what it means for π to be a **probability measure** on S.

(b) Give an example of a probability measure π on S.

(c) Can you find a subset $T \subset S$ so that the **restriction measure** r on T is **not** a probability measure? If not, go back and find a second example for part **(b)** so that you can find such a subset T.

(d) Find an example of a probability measure π on S and a proper subset $T \subset S$ so that the restriction measure r **is** a probability measure on T.

Problem 5 (section 1.4.3 from my notes) Let S be a measure space with probability measure $\pi : \wp(S) \to [0, 1]$.

(a) Given a subset $T \subset S$ define the **restriction measure** $r = r_T$ and show r is a measure.

(b) Given a subset $T \subset S$ define the **conditional probability measure** $\rho = \rho_T$ and show ρ is a measure.

Problem 6 (section 1.4.3 in my notes) Let $S = \{\omega_1, \omega_2, \ldots, \omega_n\}$ be a set with $\#S = n < \infty$. Consider $\# : \mathcal{P}(S) \to [0, \infty)$, i.e., the cardinality of sets given by the number of elements in the set.

(a) Show # is a measure.

(b) To which probability measure is # related and how?

Problem 7 (section 1.4.3 in my notes) Let

$$S = \{\text{one}, \text{two}, \text{three}, \text{four}, \text{five}, \text{six}\}.$$

Consider the uniform probability measure π on S. Let

$$A = \{\text{one}, \text{three}, \text{five}\}.$$

and

$$B = \{\text{four}, \text{five}, \text{six}\}.$$

(a) Calculate $\rho_B(A)$ and $\rho_A(B)$.

(b) Explain how Bayes' rule relates the two conditional probabilities you found in part **(a)**.

Problem 8 (section 1.4.3 in my notes) Let

$$S = \{\text{one}, \text{two}, \text{three}, \text{four}, \text{five}, \text{six}\}.$$

Consider the uniform probability measure π on S. Let

$$A = \{\text{one}, \text{three}, \text{five}\}.$$

and

$$B = \{\text{one}, \text{two}\}.$$

(a) Calculate $\rho_B(A)$ and $\rho_A(B)$.

(b) Explain how Bayes' rule relates the two conditional probabilities you found in part **(a)**.

Problem 9 (Problems 7 and 8 above) Consider the value

$$\pi(A \cap B)$$

in each of Problem 7 and Problem 8 above.

(a) In which problem(s) is the value $\pi(A \cap B)$ the same as $\pi(A)\pi(B)$?

(b) How would you describe the meaning of what you found in part **(a)** above?

Problem 10 (Problems 7 and 8 above and section 1.4.3 in my notes) Let S be the measure space with the uniform probability measure π under consideration in Problems 7 and 8 above, and consider the real valued function $x : S \to \mathbb{R}$ by

$$\begin{aligned} x(\text{one}) &= 1 \\ x(\text{two}) &= 2 \\ x(\text{three}) &= 3 \\ x(\text{four}) &= 4 \\ x(\text{five}) &= 5 \\ x(\text{six}) &= 6. \end{aligned}$$

For each of Problem 7 and Problem 8 above calculate the following:

(a) The integral of x over A with respect to π.

(b) The integral of x over B with respect to π.

(c) The integral of x over S with respect to π.

(d) The integral of x over A with respect to ρ_A.

(e) The integral of x over B with respect to ρ_A.

(f) The integral of x over S with respect to ρ_A.

(d) The integral of x over A with respect to ρ_B.

(e) The integral of x over B with respect to ρ_B.

(f) The integral of x over S with respect to ρ_B.