Assignment 1: Sets, Functions, Measures
Due September 12, 2023

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Problem 1 (From section 1.4.1 of my notes) Let $\mathcal{F}$ denote a family of sets.

(a) Prove De Morgan's law

$$\left(\bigcup_{A\in\mathcal{F}} A\right)^c = \bigcap_{A\in\mathcal{F}} A^c.$$

(b) (counting) If $\#\mathcal{F} = m < \infty$ so that $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ and

$$\#A_j = n_j \qquad \text{for} \qquad j = 1, 2, \ldots, m,$$

then find the cardinality of the Cartesian product

$$\prod_{A\in\mathcal{F}} A.$$

Problem 2 (From section 1.4.2 of my notes) Let $x : R \to S$ be a function.

(a) State precisely what it means for x to be injective.

(b) Show x is injective if and only if there exists a function $y : S \to R$ such that $y \circ x = \text{id}_R$.

(c) State precisely what it means for x to be surjective.

(d) Show x is surjective if and only if there exists a function $y : S \to R$ such that $x \circ y = \text{id}_S$.

Problem 3 (From section 1.4.3 of my notes) Given a (baby) measure μ on a set $S = \{\omega_1, \omega_2, \ldots, \omega_n\}$, explain why the restriction

$$\mu\Big|_{\Omega} \qquad \text{to} \qquad \Omega = \{\ \{\omega_1\}, \{\omega_2\}, \ldots, \{\omega_n\}\ \}$$

is **not** a measure.

Problem 4 (Exercise 1.4.3 from my notes) Let $S = \{\omega_1, \omega_2, \ldots, \omega_n\}$ be a set with n elements.

(a) Define what it means for π to be a **probability measure** on S.

(b) Give an example of a probability measure π on S.

(c) Can you find a subset $T \subset S$ so that the **restriction measure** r on T is **not** a probability measure? If not, go back and find a second example for part **(b)** so that you can find such a subset T.

(d) Find an example of a probability measure π on S and a proper subset $T \subset S$ so that the restriction measure r **is** a probability measure on T.

Problem 5 (section 1.4.3 from my notes) Let S be a measure space with probability measure $\pi : \wp(S) \to [0, 1]$.

(a) Given a subset $T \subset S$ define the **restriction measure** $r = r_T$ and show r is a measure.

(b) Given a subset $T \subset S$ define the **conditional probability measure** $\rho = \rho_T$ and show ρ is a measure.

Problem 6 (section 1.4.3 in my notes) Let $S = \{\omega_1, \omega_2, \ldots, \omega_n\}$ be a set with $\#S = n < \infty$. Consider $\# : \mathcal{P}(S) \to [0, \infty)$, i.e., the cardinality of sets given by the number of elements in the set.

(a) Show # is a measure.

(b) To which probability measure is # related and how?

Problem 7 (section 1.4.3 in my notes) Let

$$S = \{\text{one}, \text{two}, \text{three}, \text{four}, \text{five}, \text{six}\}.$$

Consider the uniform probability measure π on S. Let

$$A = \{\text{one}, \text{three}, \text{five}\}.$$

and

$$B = \{\text{four}, \text{five}, \text{six}\}.$$

(a) Calculate $\rho_B(A)$ and $\rho_A(B)$.

(b) Explain how Bayes' rule relates the two conditional probabilities you found in part **(a)**.

Problem 8 (section 1.4.3 in my notes) Let

$$S = \{\text{one}, \text{two}, \text{three}, \text{four}, \text{five}, \text{six}\}.$$

Consider the uniform probability measure π on S. Let

$$A = \{\text{one}, \text{three}, \text{five}\}.$$

and

$$B = \{\text{one}, \text{two}\}.$$

(a) Calculate $\rho_B(A)$ and $\rho_A(B)$.

(b) Explain how Bayes' rule relates the two conditional probabilities you found in part **(a)**.

Problem 9 (Problems 7 and 8 above) Consider the value

$$\pi(A \cap B)$$

in each of Problem 7 and Problem 8 above.

(a) In which problem(s) is the value $\pi(A \cap B)$ the same as $\pi(A)\pi(B)$?

(b) How would you describe the meaning of what you found in part **(a)** above?

Problem 10 (Problems 7 and 8 above and section 1.4.3 in my notes) Let S be the measure space with the uniform probability measure π under consideration in Problems 7 and 8 above, and consider the real valued function $x : S \to \mathbb{R}$ by

$$\begin{aligned} x(\text{one}) &= 1 \\ x(\text{two}) &= 2 \\ x(\text{three}) &= 3 \\ x(\text{four}) &= 4 \\ x(\text{five}) &= 5 \\ x(\text{six}) &= 6. \end{aligned}$$

For each of Problem 7 and Problem 8 above calculate the following:

(a) The integral of x over A with respect to π.

(b) The integral of x over B with respect to π.

(c) The integral of x over S with respect to π.

(d) The integral of x over A with respect to ρ_A.

(e) The integral of x over B with respect to ρ_A.

(f) The integral of x over S with respect to ρ_A.

(d) The integral of x over A with respect to ρ_B.

(e) The integral of x over B with respect to ρ_B.

(f) The integral of x over S with respect to ρ_B.