Normal Distribution Derivation

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Suppose I were throwing darts onto a dartboard, trying to hit the center of the target. Consider a Cartesian coordinate system over the dartboard, with the center at (0, 0). Every dart I throw is going to land at some random position (x, y), and the values of x and y represent how far off the target I was in their respective directions. Now, I want to find the joint probability distribution $p(x, y)$ that models my dart throws.

Let's start with two reasonable assumptions about the way we play darts. The first is that the probability density at each point of the dartboard, depends only on its distance from the center. For example, if we were to rotate the dartboard in any way, the distribution would stay the same. In other words, the distribution $p(x, y)$ we're trying to find is rotationally symmetric. Then, we can express our distribution as

$$p(x, y) = f(r) = f(\sqrt{x^2 + y^2}),$$

where r is the distance between the point (x, y) and the origin, and f is some single-variable function. The second assumption is that the distributions of x and y are independent, meaning that how much I miss left and right doesn't affect how much I miss up and down, and vice versa. Let $g(x)$ be the distribution of x and $h(y)$ be the distribution of y. So,

$$p(x, y) = g(x)h(y).$$

But from our first assumption of rotational symmetry, the distribution of each axis must be the same, so $g = h$. By combining the two assumptions we get

$$f(r) = f(\sqrt{x^2 + y^2}) = g(x)g(y).$$

By setting $y = 0$, we see that

$$f(x) = g(x)g(0),$$

where $g(0)$ is a constant, i.e., $f(x)$ is proportional to $g(x)$. Then, we can substitute $f(\sqrt{x^2 + y^2})$ with $g(\sqrt{x^2 + y^2})g(0)$ to get

$$g(\sqrt{x^2 + y^2})g(0) = g(x)g(y).$$

This is a functional equation where we are solving for g. We start by dividing both sides by $g(0)^2$:

$$\frac{g(\sqrt{x^2 + y^2})}{g(0)} = \frac{g(x)}{g(0)}\frac{g(y)}{g(0)}.$$

To make this expression cleaner, let's define the function $j(x) = \frac{g(x)}{g(0)}$:

$$j(\sqrt{x^2 + y^2}) = j(x)j(y)$$

and the function $\ell(x) = j(\sqrt{x})$, so that $\ell(x^2) = j(x)$:

$$\ell(x^2 + y^2) = \ell(x^2) + \ell(y^2).$$

What we are left with is called the Cauchy's exponential functional equation: $\ell(a) + \ell(b) = \ell(a + b)$. I will show that over the rational numbers, $\ell(x)$ can only be the family of functions b^x for some constant b. Let's first consider integers. For $x \in \mathbb{Z}$, we see that

$$\begin{aligned}
\ell(x) &= \ell(\underbrace{1 + 1 + \cdots + 1}_{x \text{ times}}) \\
&= \ell(1)\ell(1)\cdots\ell(1) \\
&= \ell(1)^x \\
&= b^x, b = \ell(1) \text{ is some constant}
\end{aligned}$$

Next, for $x \in \mathbb{Q}$, we know $x = \frac{p}{q}, p, q \in \mathbb{Z}$:

$$\begin{aligned}
\ell(\underbrace{\frac{p}{q} + \frac{p}{q} + \cdots + \frac{p}{q}}_{q \text{ times}}) = \ell(p) &= \ell(\frac{p}{q})^q \\
\ell(p)^{1/q} &= \ell(\frac{p}{q}) \\
b^{p/q} &= \ell(\frac{p}{q})
\end{aligned}$$

If we make the assumption that our function is continuous, then $\ell(x)$ has to be the exponential function for $x \in \mathbb{R}$. Since b is just a constant, we can also write $\ell(x) = e^{ax}$ for some constant a. So, we know $j(x) = e^{ax^2}$ and $g(x) = g(0)e^{ax}$. Since $g(x)$ is a probability distribution and the area under it is 1, we must have $a < 0$. Let $a = -b^2$ and the constant $c = g(0)$. Then, we have that

$$c \int_{-\infty}^{\infty} e^{-b^2x^2}\, dx = 1$$

If we substitute $u = bx, du = b\, dx$, then we get the Gaussian integral which evaluates to $\sqrt{\pi}$

$$\begin{aligned}
&\frac{c}{b} \int_{-\infty}^{\infty} e^{-u^2}\, du = 1 \\
&\frac{c}{b}\sqrt{\pi} = 1 \\
&b^2 = \pi c^2
\end{aligned}$$

So, our expression becomes $g(x) = c \cdot e^{-\pi c^2 x^2}$. To make this look like our normal distribution formula, let's try to express c in terms of the variance σ^2. Since our distribution is symmetric about 0, we know the mean $\mu = 0$:

$$\sigma^2 = c \int_{-\infty}^{\infty} x^2 e^{-\pi c^2 x^2}\, dx$$

We do integration by parts with

$$u = x,\ du = dx,\ dv = x \cdot e^{-\pi c^2 x^2}.$$

Find v via integration by substitution:

$$v = \frac{1}{-2\pi c^2} e^{-\pi c^2 x^2}$$

Now the variance is

$$\sigma^2 = c(\frac{x}{-2\pi c^2} e^{-\pi c^2 x^2})\Big|_{-\infty}^{\infty} - c \int_{-\infty}^{\infty} \frac{1}{-2\pi c^2} e^{-\pi c^2 x^2}\, dx$$

The first term becomes $0 - 0 = 0$, and by repositioning the constants of the second term, we get

$$\sigma^2 = -\frac{1}{-2\pi c^2} \int_{-\infty}^{\infty} c e^{-\pi c^2 x^2}\, dx$$

But now we are integrating our distribution, so the integral must evaluate to 1, and we get

$$\sigma^2 = \frac{1}{2\pi c}$$
$$c = \frac{1}{\sigma\sqrt{2\pi}}$$

Now we can write our distribution $g(x)$ as

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x}{\sigma})^2}$$

This is exactly the PDF for the normal distribution when the mean is 0, if we add one more parameter μ to shift our distribution left and right:

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2},$$

we get our full formula parameterized by the standard deviation and the mean.

Exercises

1. Given the formula of the normal distribution, prove that the mean is μ
2. The Gaussian integral is a non-elementary integral. What are elementary functions? Give two more examples of non-elementary integrals.
3. Show that the Gaussian integral:
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$
evaluates to $\sqrt{\pi}$. Hint: Consider the function $f(x, y) = e^{-(x^2+y^2)}$.
4. Show that the inflection points of the PDF are at $x = \mu + \sigma$ and $\mu - \sigma$.
5. Show that the area under the PDF is 1.