1 Deciphering The Monty Hall Problem

The Scenario

Welcome to Monty Hall's game show where you can either strike it rich... or walk out with nothing. The premise is simple - behind 3 doors, there are 2 goats (duds) and 1 supercar (treasure). You choose one door at random in hopes of winning the supercar. Upon making your choice, Monty, who knows what is behind each door, then looks at the two unchosen doors and removes one of the doors that is a dud. You are then presented with the following choice: you can stay with your original door choice, or you can switch doors into the door that Monty did not cross out. In order to maximize the chances of winning the supercar, is there a reason to switch?

Incorrect Assumptions

This is a question of independent events as the game show progresses. Contrary to popular belief, after Monty Hall removes one of the doors that holds a goat, the chance of winning the supercar is *not* 50/50 even though there are two choices. In this case, there is a gain of knowledge that skews the results into the act of switching's favor. In other words, when Monty opens the door and reveals the goat, the $\frac{1}{3}$ chance of that door gets passed onto the unopened door.

Brute Force Method

Perhaps you are burnt out and want to quickly finish your homework and the Monty Hall problem is the last problem on your homework. You can always brute force the situation by drawing out all of the cases and then calculating the probabilities manually. The tables should resemble the following:

- Assuming that you will always pick door 1 (it doesn't really matter because it is a $\frac{1}{3}$ chance regardless)
- Assume the yellow color represents the door Monty is removing after you decide on your initial $\frac{1}{3}$ door.

Door 1	Door 2	Door 3	Don't Switch	Switch
Goat	Goat	Supercar	Lose	Win
Goat	Supercar	Goat	Lose	Win
Supercar	Goat	Goat	Win	Lose

Mathematical Proof

Assume that you are always going to pick door 1
Let:

- A_i = event that supercar is behind door i
- C_i = event that Contestant chooses door i
- M_i = event that Monty opens door i (recall that he never opens that door that you choose or the door with the supercar)

Initial Events:

$P(A_i) = \frac{1}{3}$

$P(C_i) = 1$ (Recall that you are always picking door 1 so 'i' might as well be 1)

Base Cases:

Given Door 3 is opened and you pick Door 1 and the supercar is behind Door 1

$$P(M_3|A_1 \cap C_1) = \frac{1}{2} \quad \text{Monty can either open door 2 or door 3}$$

Given Door 3 is opened and you pick Door 1 and the supercar is behind Door 2

$$P(M_3|A_2 \cap C_1) = 1 \quad \text{Monty can only open door 3}$$

Given Door 3 is opened and you pick Door 1 and the supercar is behind Door 3

$$P(M_3|A_3 \cap C_1) = 0 \quad \text{Monty can never open door 3; he can only open door 2}$$

Bayes Theorem Extrapolation

$$P(A_2|M_3 \cap C_1) = \frac{P(M_3|A_2 \cap C_1) \cdot P(A_2)}{P(M_3, C_1)}$$

$$\begin{aligned}
P(M_3|C_1) &= P(M_3|A_1 \cap C_1) \cdot P(A_1) + P(M_3|A_2 \cap C_1) \cdot P(A_2) + P(M_3|A_3 \cap C_1) \cdot P(A_3) \\
&= \left(\frac{1}{2}\right) \cdot \left(\frac{1}{3}\right) + 1 \cdot \left(\frac{1}{3}\right) + 0 \cdot \left(\frac{1}{3}\right) \\
&= \left(\frac{1}{6}\right) + \left(\frac{2}{6}\right) \\
&= \frac{1}{2}
\end{aligned}$$

Therefore:

$$
\begin{aligned}
P(A_2|M_3 \cap C_1) &= \frac{(P(M_3|A_2 \cap C_1) \cdot P(A_2))}{P(M_3, C_1)} \\
&= \frac{1 \cdot \left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)} \\
&= \frac{2}{3}
\end{aligned}
$$

In other words, given that Monty has opened Door 3, which implies that the supercar is not in Door 3, the probability that the supercar is in Door 2 is $\frac{2}{3}$ and by deduction, the probability that the supercar is in Door 1 (the door that you pick) is $\frac{1}{3}$.