Liar’s Dice Analysis

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Description of Liar’s Dice

Liar’s Dice has been played for a long time and came to be even more known after appearing in the film Pirates of the Caribbean. The game is played traditionally between 2-5 players and consists of every player getting 5 dice and “rolling” * them inside a cup without showing to their competitor the results. After rolling, the game itself starts with the players trying to predict the number of dice with a face on the table (dice with face “1” as joker worth any other side). A prediction consists of saying a number of dice that the table has on a specific number (“2”, “3”, “4”, “5”, “6”). Starting by the second player clockwise to do the prediction he/she can either:

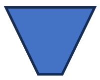
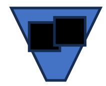
1. challenge the prior statement. Then everyone shows their hidden dice for counting purposes as the predicted part of the round finished.
    a. If on the table there is equal or more dice of the number (includes also “1”, because “1” is a joker and can be any “face” value) prior player had predicted then the challenger (second player) loses 1 dice for next round.
    b. If on the table there is less than the number of dice predicted by the prior player, the prior player loses a dice.
2. Predict a higher number of dice and/or “face.”
    a. increase the number of dice predicted to have a specific face. This means there is no requirement for what face value it needs to be. (For instance, prior prediction is 3 dice of face “6” faces; new prediction can be 4 dice with face “2”, 4 dice with face “3”, 4 dice with face ”4”, 4 dice with face “5”, 4 dice with face “6”, 5 dice with face “2”, 5 dice with face “3”, …)
    b. increasing the “face” value, the number of dice must stay the same or increase. (For instance, prior predict 3 “3”; new predict can be 3 dice with face “4”, 3 dice with face “5”, 3 dice with face “6” but NOT 3 dice with face “2”, if increased the number of dice rule 2a already has the options listed already).

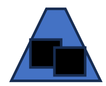
In other words, the prediction order from “easier” to more “difficult” is 1 dice with face “2” 1 dice with face “3”, 1 dice with face “4”, 1 dice with face “5”, 1 dice with face “6, 2 dice with face “2”, 2 dice with face “3”, 2 dice with face “4”, 2 dice with face “5”, 2 dice with face “6” and so on.

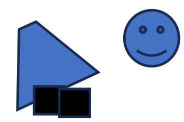
Extra note the player might challenge because if he does not challenge a prediction, he/her thinks is wrong he will have to make a prediction that most probably will have even lower chances of happening giving him a higher chance of losing a die and eventually losing the game.

End game: The game ends when there is only one person with dice left.

*In this game there is a limited space instead of what you normally see of people throwing the dice as rolling, the dice is inside the cup, and you swing it upside down carefully (see schematic below). Also, for this schematic the cup is transparent, but it is not in the real game.

Cup  Die  1- Dice inside Cup 

2- turn the cup with dice (or die inside) 

3- Don't show dice (or die) to opponent and take a pick inside (similar to poker and card games) 

4-After every player has completed steps 1 and 2 as well as normally also 3 (theoretically you can play the game without seeing your prediction. though it will reduce your chances of winning), the 1st player can start giving his prediction. Just emphasizing again, the player only knows his/her dice, the only information he/she has on the other player dice are from eventually prediction made by them.

Scenario for analysis

Initially, is going to analyze the initial scenario (all people with 5 dice) and 5 people all of which are playing optimally predictions with no irrational moves. Also, all the dice are considered fair dice. That is a model for predicting predicts considering everyone is optimizing their predicts and values for the highest predicts that make them have the expectation of being right (50%>) and if there is no such option pick the one with highest possibility of not making the player lose a die.

1st person analysis

For initial predict the player knows his 5 dice and has 5 options for his dice: 5 of kind, 4 of a kind 3 of a kind, 2 of a kind. We will call this "face" with most of a kind by L (L can be "2", "3", "4", "5" or "6") and the cardinality by N_1 And 20 dice not known. By integrating the following, where S is all the option of the 25 dice represented by the measure S= $\{“1”,”2”,”3”,”4”,”5”,”6”\}^{25}$ and x is the dice probability with respect to B. B has value 1 for S element=L or 1and 0 for all other values.

$$x_* = \int_S X = \frac{1}{6} \times 20 + \frac{1}{6} \times 20 + 0 = 6.67$$

This means that the 1st person predict should be 6 + N_1.

2nd player analysis

He/She knows the prior predict and so what is the maximum number of dice of the person.

In this case the not known dice are 20-N_1 (5-N1 from player1, 5 from players3,4,5) (not L)
Similarly resulting:

$$x_* = \int_S X = \frac{1}{3} \times 20 - \frac{1}{3} \times N1 + \frac{1}{2} \times N1 = 6.67 - \frac{1}{6} \times N1$$

This can be separated to 4 scenarios with N_2 representing the cardinality of the dice he has the most:

1. He/She has 3 or more dice with face L or "1", this cardinality is represented by N_2. In this scenario adding N_2-2 to the number offered by the prior player will result in more in a probable outcome to be right.
2. He/She has N_1*7/6-0.67 or more dice of another face than L called T (including "1") and less than 3 of L and "1"**. In general this just means 1 more dice than before. In this case independent of face L and T player2 has a positive outcome if he offers 6.67-1/2*N_1+N_2. 6.67 is number of dice of other players you don't know (5-Very similar to 1st person move, but with adapted because the 1st predict gives clues about the dice of the 1st person.).
3. He/She has N1*7/6+0.33 or more dice of another face than L called T, with L a bigger face value than 3 (following predicting rule 2). This is just option 2 with 1 less value increase.
4. He/She has less than N_1*7/6-0.67 dice of another face than L called T (including "1") and less than 3 of L and "1". This is an interesting case because there is no option with a more than 50% outcome and one less than probable outcome has to be chosen.
   a. The 1st idea would be to challenge the prior statement just to minimize the negative percentage, but actually doing a prediction might actually increase the chances of the player because of Options 1 and 2.
   b. 2nd option is to predict a less than probable option (that still is the most probable for you) and get Option 4 with same result as Option 1, 2, 3. Generating a slight increase in probability that the next player does not challenge your prediction although you are wrong. In this case you select the option with the least amount of difference in terms of expected value.

   This results in option 4b been the preference.

**This is based on the inequality of 6.67-1/6*N_1+N_2 >= 6+ 1 + N_1 To allow the to have 1 more dice than the prior guess

3rd Players analysis

The number of dice not known any information on is 15-N1-N2. Adding to that it is known that N1 dice have value "L" or "1" generating 1/2N1 dice in poll. Now the tricky part is generating an estimate if the prediction of player 2 is due to options 1,2,3 or option 4. This is important because there is N2*1/2 dice that can be "1" for options 1,2,3 while in option 4 that is not an option.

For this it is necessary to calculate what is the probability of option 4 versus option 1,2 or 3.

Calculation true probability in terms of options:

Option 1 for has a probability of happening with the minimum prediction (assuming that the player does not predict over their expected value to lower probability – this means that if you have not the even 3 of "L" or "1" player 2 will NOT predict 4 of "L" or "1" as this will start skewing the result to any prediction that is offered is more probable to be a lie than true). This is probability of having exactly 3 options right:

$$E(X) = \binom{5}{3} \times [P(3L) + P(2L + "1") + P(L + "1")]$$

$$= \binom{5}{3} \times \left[\frac{4}{9} \times \frac{1}{6}^3 + \frac{3}{6^3} \times \frac{4 + 4 \times 3}{6^2} + \frac{3}{6^3} \times \frac{4 \times 3}{6^2}\right] = 13\%$$

A faster but slightly off calculation would be

$$E(X) = \binom{5}{3} \times \frac{1}{3}^3 \times \frac{2}{3}^2 = \frac{40}{243} = 16.4\%$$

(Please note this result is slightly overestimated. I say this because there is inside the calculated numbers that there is 4 of another face M especially if the values contributing to L are "1". But, also note that apply this to a game needs to have easy calculations so approximations make sense, see appendix for more in depth calculations**)

Option 2 is the call with minimum number of dice with face T. This happens for 5 dice whenever there are 4 dice with same two options of faces T or "1".

$$E(X) = \binom{5}{4} \times \frac{1}{6} \times \frac{1}{3}^2 \times \frac{12}{36} = 3.1\%$$

(Please note this result is slightly overestimated, if needed for the final project draft I can be more specific with values. I say this because there is inside the calculated numbers that there is 4 of another face M specially if the values contributing to T are "1")

Option 3 for has a probability of happening with the minimum prediction (assuming that the player does not predict over their expected value to lower probability – this means that if you have not the even 3 of "L" or "1" player 2 will NOT predict 4 of "L" or "1" as this will start skewing the result to any prediction that is offered is more probable to be a lie than true). This is probability of having exactly 3 options right:

$$EE(X) = \binom{5}{3} \times [P(3L) + P(2L + "1") + P(L + "1")]$$

$$= \binom{5}{3} \times \left[\frac{4}{9} \times \frac{1}{6}^3 + \frac{3}{6^3} \times \frac{4 + 4 \times 3}{6^2} + \frac{3}{6^3} \times \frac{4 \times 3}{6^2}\right] = 13\%$$

Faster but slightly off calculation

$$E(X) = \binom{5}{3} \times \frac{1}{3}^3 \times \frac{2}{3}^2 = \frac{40}{729} = 16.4\%$$

(Please note this result is slightly overestimated, if needed for the final project draft, I can be more specific with values. I say this because there is inside the calculated numbers that there is 4 of another face M specially if the values contributing to T are “1”)

Option 4 is in this minimum case when there are not 3 dice of same number. This means that the best prediction depends on which dice you have. If the dice have faces “L” or a face bigger than “L” this has a bigger probability given by Options 1 or 3, thus they will be better to hide the true value.

$$E(X) = \frac{5! + \binom{5}{2} \times 5 \times 4 \times 3 \times 2}{6^5} = \frac{720 + 1200}{7776} = 17\%$$

Specifically, when comparing option 2 there is also the possibility of 3 dice of face t or “1”. This means. This adds 2.6% for that option.

This value was calculated using permutations for the few options available (5 different values – 5! Options for each with one dice face value “excluded” or 2 of the same face value, but no one’s generating 5!/2 permutations options and 4 options for the value been used twice and 1 for the face value not being used that is not “1”).

So comparatively there if the prediction is false for options 2 is 17.1%/20% = 85.5% while for option 1 and 3 is roughly 17%/30%=57%. Note this was made illustratively with N_2 equal to 3 for options 1 and 3 as well as 4 for option 2 because they are the most probable outcome for those predictions. This is especially important because if N_2 is higher than those numbers (it cannot be lower for the predicting) the probability that the predicting is not a real value becomes higher than 85.5% and 57% respectively.

Now the same process made for player 2 can be made for player 3 with the caveat that the options N_2 have to be multiplied by the probability that the results are right true or not.

This is also important to show how much more improbable there is that it should even get to the 4th player in the game.

**In depth calculations:

Two “1” and 1 “L” -> 4*3*2/6^3 = 24/216, P (two “1”, 1”L) = 3/6^3*24/216

One “1” and 2 “L” -> (4*1*3+4*3*2)/6^3 = 36/216, P (two “1”, 1”L) = 3/6^3*36/216

Three “L” -> (2/3)^3 = 8/27, 8/27*1/6^3

Total sum: 1.37*10^-3

Result 1.37*10^-3*(63)= 2.7%

Conclusion

This gives an idea that 13.5%*(Magnitude of L-1) + 6-(Magnitude of L) *3.1% = [25.9%-70.6%] Percentage of being right with a one number of dice increase compared to [5%-59%] of a “lie” (type 4) with higher probability while the rest is probability of higher prediction. This number will still increase, but in round 2 the probability the player will lose is still small but starting with the 3rd player probabilities with other dice faces start to decrease with a higher number required. This should increase exponentially as only 13.5% of L stays with that percentage. While the rest decreases even lower to 5% or less. Making the probability of “lie” increasing even more especially if the predict is adding to the already proposed values. So, as a rule of thumb there will not be more than 2 rounds, most probably the game will end at the end of the 1st round with the “lie” prediction.

*Magnitude of L is the face value of L

4th to infinite player/round analysis

Similar analysis as player 3, considering N_3 equivalent to N_2 in terms of options.

Problems:

1) Do a similar process for 3 dice for each of the 5 players to calculate the low and high ranges of the 2nd prediction being “lie”. Do it with the more precise method.
2) Now consider yourself lucky (or a player that has read this document). You were able to stay with 5 dice while all other players lost 3. Now simulate your position and prediction as the 1st, 2nd, and 3rd players.
    a. In the 1st player situation include how the 2nd player chance of sending out a “lie” prediction is higher and by how much

b. For the 2nd player, show how irrelevant the 1st prediction might be.
c. For 3rd player position do like problem 1

3) Now you are at the other end, you have read but not paid attention to this report. You lost 3 dices\ straight away. Do similar analysis to problem 2, but for just 2nd and 3rd positions and show how your position is unfavorable now.
4) Extra rule: if there are only 2 dices\ left instead of having a prediction for only two dice instead of predicting for the possible dice faces, there is a prediction on the sum of the faces of the dice. Same rules about challenges, but now you can only increase in number of the sum or challenge. Analyze this case and what should be your prediction as 1st player, and as 2nd player, that is if the system for this report is kept. Is there a definitive winning position?