# Chebyshev's Inequality

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Hanna Glamm was interested in the sequence of MDFs given by

$$\delta_n(\omega) = |\omega|^n \; \chi_{[-a,a]}$$

for $n = 1, 2, 3, \ldots$ and $a = a(n)$ is an appropriate constant so that

$$\int_{-a}^{a} \delta_n(\omega)\, d\omega = 1.$$

She was interested in how well the associated central masses

$$M_n = \int_{\sigma}^{\sigma} \delta_n(\omega)\, d\omega$$

performed in Chebyshev's inequalty as $n$ tends to infinity where $\sigma = \sigma(n)$ is the standard deviation associated with $\delta_n$. Chebyshev's inequality says

$$\int_{k\sigma}^{k\sigma} \delta(\omega)\, d\omega \geq 1 - \frac{1}{k^2}$$

where $\sigma$ is the standard deviation satisfying

$$\sigma^2 = \int_{-\infty}^{\infty} (\omega - \mu)^2\, \delta(\omega)\, d\omega$$

and $\mu$ is the mean given by

$$\mu = \int_{-\infty}^{\infty} \omega\, \delta(\omega)\, d\omega.$$

The quantity $M_n$ gives the left side in Chebyshev's inequality when $k = 1$ and the right side of the inequality is zero. Thus, the question is how close does $M_n$ get to zero. Hanna's preliminary conclusion via numerical calculation was that $M_n$ tends to zero "in a strange way." I'm pretty sure what she was seeing was a round off phenomenon which occurred for me around $n = 260$. I believe that $M_n$ decreases with $n$ (also from numerical calculation though I might be able to prove it). What I can do is calculate the limit. Here is the explicit calculation:

Each of the densities $\delta_n$ has mean zero, so the variance is given simply by

$$\sigma^2 = \int_{-\infty}^{\infty} \omega^2 \, \delta_n(\omega) \, d\omega.$$

To make this and the other calculation(s) we need to know the value of $a$ determining the range. Specifically, we need $a = a(n)$ for which

$$\int_0^a \omega^n \, d\omega = \frac{1}{2}.$$

That is,

$$\frac{a^{n+1}}{n+1} = \frac{1}{2} \qquad \text{or} \qquad a = \left(\frac{n+1}{2}\right)^{1/(n+1)}.$$

Thus, we have

$$\begin{aligned} \sigma^2 &= 2\int_0^a \omega^2 \, \omega^n \, d\omega \\ &= \frac{2}{n+3} a^{n+3}. \end{aligned}$$

Consequently,

$$\sigma = \sqrt{\frac{2}{n+3}} a^{\frac{n+3}{2(n+1)}},$$

and

$$
\begin{aligned}
M_n &= 2\int_0^\sigma \omega^n \, d\omega \\
&= \frac{2}{n+1}\sigma^{n+1} \\
&= \frac{2}{n+1}\left(\frac{2}{n+3}\right)^{\frac{n+1}{2}} a^{\frac{(n+3)(n+1)}{2}} \\
&= \frac{2}{n+1}\left(\frac{2}{n+3}\right)^{\frac{n+1}{2}} \left(\frac{n+1}{2}\right)^{\frac{n+3}{2}} \\
&= \left(\frac{n+1}{n+3}\right)^{\frac{n+1}{2}}.
\end{aligned}
$$

Therefore,

$$
\ln M_n^2 = (n+1)\ln\left(\frac{n+1}{n+3}\right) = \frac{\ln\left(\frac{m}{m+2}\right)}{1/m}
$$

where $m = n+1$. The numerator

$$
\ln\left(\frac{m}{m+2}\right)
$$

tends to zero since $m/(m+2)$ tends to one. Also the denominator $1/m$ tends to zero, so we can try L'Hopital's rule:

$$
\frac{\frac{d}{dm}[\ln m - \ln(m+2)]}{-1/m^2} = -m^2\left(\frac{1}{m} - \frac{1}{m+2}\right) = -\frac{2m^2}{m(m+2)}.
$$

This quantity tends to $-2$ as $m \nearrow \infty$. Thus,

$$
\lim_{n\nearrow\infty} M_n = \left(e^{-2}\right)^{1/2} = \frac{1}{e}.
$$

This is more or less compatible with the numerical calculation, though I'm not entirely happy with that as I get

$$
M_{200} \doteq 0.369702, \qquad M_{250} \doteq 0.36934 \qquad \text{and} \qquad \frac{1}{\sqrt{e}} \doteq 0.367879.
$$

I guess $1/e$ is correct however. I guess Mathematica is just having trouble computing the value due to excessive round off error or something like that.