Two-card Poker Hands

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Apparently in "Texas hold 'em" poker players are initially dealt two-card hands and subsequently participate in a round of betting based on the two cards each player can see in his own hand but with no knowledge of the two cards in the hands of other players. Victor Henriksson who is studying/considering strategies for this first round of play is interested in classifying and counting the number of possible two-card hands. He has suggested a classification involving 6 distinct kinds of hands as follows

1. pair: two cards of the same rank
2. connector: ranks differing by exactly one, e.g., ranks 2 and 3
3. one-gap: ranks differing by exactly two, e.g., ranks 2 and 4
4. two-gap: ranks differing by exaclty three, e.g., ranks 2 and 5
5. three-gap: ranks differing by exactly four, e.g., ranks 2 and 6
6. nothing hands: none of the above

There are a total of

$$\binom{52}{2} = 26(51) = 1326$$

possible two-card hands, and the following are the counts for each type in Victor's classification:

1. pair: Generally speaking we will count according to the rank of the cards in the pair and then consider the suits of those cards. For a pair there is only one rank involved from among the ranks

$$A\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ J\ Q\ K, \tag{1}$$

so we can say this is the "rank of the hand." The count of pair hands is

$$(13 \text{ choices for the rank}) \times \binom{4}{2} = 13(6) = 78$$

Note: The factor

$$6 = \binom{4}{2}$$

corresponds to the number of ways to assign suits to the two cards. The suits must be chosen from among 4 suits and must be different. The order of the choice does not matter, as the ranks are the same.

2. connector: For connector hands and all the remaining hands counted below (that are not pairs) the following hold:

   (a) The hand will have a unique lower rank card and a unique higher rank card, so we can call these the "low card" and the "high card."

   (b) It is possible for the two cards to have the same suit or to have different suits. We designate two-card hands with the same suit (or flush hands) as "suited," and hands with cards of two different suits as "off-suited."

   For a connector hand, once the rank of the low card is chosen, the rank of the high card is determined. Also, since ace A can count as high or low, any card may be chosen as the low card. Thus, the count for connector hands is as follows

   (i) suited:

   $$(13 \text{ choices for the low card}) \times (4 \text{ choices for the suit}) = 52.$$

   (ii) off-suited:

   $$(13 \text{ choices for the low card}) \times P(4, 2) = 13(12) = 156.$$

   Note: Here the ranks of the two cards are different and because of this, the order in which the suits are assigned matters. That is, there are

   $$\begin{aligned}&(4 \text{ choices for the suit of the low card})(4 \text{ choices for the suit of the high card})\\ &\qquad = P(4, 2) = 12\end{aligned}$$

   choices for the way the suits can be assigned to the cards in the (connector) hand. This principle continues to apply to other hands below.

3. one-gap: The list of cards in (1) can be helpful in counting the number of choices for the low card. Notice that K is not a possible choice of low card for a one-gap hand, so there are 12 choices for rank of the low card instead of 13. Again, once the rank of the low card is chosen, the rank of the high card is determined. The count for connector hands is as follows

   (i) suited:

$$(12 \text{ choices for the low card}) \times (4 \text{ choices for the suit}) = 48.$$

   (ii) off-suited:

$$(12 \text{ choices for the low card}) \times P(4,2) = 12(12) = 144.$$

4. two-gap: The principles of counting here are the same. Both K and Q are excluded from the list of low cards.

   (i) suited:

$$(11 \text{ choices for the low card}) \times (4 \text{ choices for the suit}) = 44.$$

   (ii) off-suited:

$$(11 \text{ choices for the low card}) \times (12 \text{ assignments of suits}) = 132.$$

5. three-gap: J Q and K are excluded from the list of low cards.

   (i) suited:

$$(10 \text{ choices for the low card}) \times (4 \text{ choices for the suit}) = 40.$$

   (ii) off-suited:

$$(10 \text{ choices for the low card}) \times (12 \text{ assignments of suits}) = 120.$$

6. nothing hands: These are a little bit trickier to count, and we could just subtract all the numbers of hands above from the total number 1326 of hands, but it is interesting (and fun) to count these hands into certain categories. First note that in this case the rank of the low card does not determine the rank of the high

card. Another way to describe nothing hands is "four or higher"-gap hands. If the low card has rank 2, then the high card may have rank 7 or higher.

Consider first the **nothing hands not containing an ace**. If the low card has rank 2, as mentioned above, then the high card can have rank 7, 8, 9, 10, J, Q, or K. There are 7 possible high cards. For low card of rank 3, there is one fewer possible high card. This continues until the low card has rank 8 at which point the high card must be a king, and there is only one possibility. Thus, there are a total of

$$1 + 2 + 3 + \cdots + 7 = \frac{7(8)}{2} = 28$$

choices for the ranks of a nothing hand that does not contain an ace. Thus, we can count these hands using the principles for assigning suits described above:

(i) suited:

$$(28 \text{ choices for the ranks}) \times (4 \text{ choices for the suit}) = 112.$$

(ii) off-suited:

$$(28 \text{ choices for the ranks}) \times (12 \text{ assignments of suits}) = 336.$$

Counting **nothing hands containing an ace** involves some additional complication. If you count an ace low, then ranks $\{A, 6\}$ and $\{A, 7\}$ should be counted, but you can't let the high card get too high, or else, the ace should be counted high and you get a gapper or connector hand. That is, $\{2, J\}$ counted as an eight-gap hand, but $\{A, 10\}$ should not count as an eight-gap hand because $\{10, A\}$ was already counted as a three-gap hand. So we get only $\{A, 6\}$, $\{A, 7\}$, ..., $\{A, 9\}$ for the possible ranks, and there are 4 such hands. Thus, we get

(iii) suited:

$$(4 \text{ choices for the ranks}) \times (4 \text{ choices for the suit}) = 16.$$

(iv) off-suited:

$$(4 \text{ choices for the ranks}) \times (12 \text{ assignments of suits}) = 48.$$

If you add up all the counts of hands listed above, you should get 1326.