Stopping an oscillator with an impulse

John McCuan

December 1, 2017

A "favorite" exercise (and exam question) of teachers of elementary ordinary differential equations (ODEs) involves the stopping of a linear oscillator with an impulse. One is given a linear second order operator with positive coefficients—the kind of operator often used to model an oscillator. Let us say this operator is

$$L[y] = my'' + \alpha y' + ky$$

where m is interpreted as the mass, k represents the Hooke's constant, and α is a damping coefficient. We are given some initial conditions setting the system in motion, say

$$y(0) = y_0 \qquad \text{and} \qquad y'(0) = y_0',$$

and we are asked to find some time (or times) T for which the system may be stopped by an impulse of a prescribed magnitude and direction. Finding the magnitude and direction of the impulse is also part of the problem. The basic idea is to use the Laplace transform of the operator with e^{-sT} modeling the unit impulse. Variants of the problem include asking for the impulse to take place at a specific pass of the mass through equilibrium, for example, the twentieth pass or simply prescribing an apriori impulse and verifying the effect via the Laplace transform method.

We will make the necessary calculation for the general parameters above and then apply the result to a specific example. There are also three exercises at the end. During the course of our solution we will verify the fact that an impulse modeled by μe^{-sT} increases the momentum of the system by an amount μ if $\mu > 0$ and decreases the momentum by an amount $|\mu|$ if $\mu < 0$. With this information, one can determine the times and values μ which are possible directly from the unforced initial value problem, without using the Laplace transform. On the other hand, it is possible to derive the possible times and coefficients μ directly using the laplace transform without considering the homogeneous problem at all. We will take a middle path based on the heuristic observation that it should only be possible to accomplish the task of stopping the motion of an oscillator with an impulse if the impulse happens when the mass passes through equilibrium.

For the homogeneous problem, the characteristic polynomial $m\lambda^2 + \alpha\lambda + k$ has roots

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4mk}}{2m}.$$

We will also freely use the normalized coefficients $p = \alpha/m$ and $q = k/m$. We assume the oscillator is underdamped, that is, $\alpha^2 < 4mk$ or equivalently $p^2 < 4q$. With this assumption we have complex roots

$$\lambda = -\frac{\alpha}{2m} \pm i\omega = -\frac{p}{2} \pm i\omega$$

where

$$\omega = \frac{1}{2m}\sqrt{4mk - \alpha^2} = \sqrt{q - p^2/4},$$

and the homogeneous solution is given by

$$y_h(t) = e^{-pt/2}(a\cos\omega t + b\sin\omega t) = \sqrt{a^2 + b^2}e^{-pt/2}\cos(\omega t - \theta)$$

where θ is determined up to a multiple of 2π by

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \qquad \text{and} \qquad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

Various information may be obtained from the homogeneous solution. First of all, the initial conditions imply

$$a = y_0 \qquad \text{and} \qquad b = \frac{y_0' + py_0/2}{\omega}.$$

The determination of b follows from the expression

$$y_h'(t) = e^{-pt/2}[-a(p/2)\cos\omega t - b(p/2)\sin\omega t - a\omega\sin\omega t + b\omega\cos\omega t].$$

In addition, the positive times at which the mass passes through equilibrium may be read off from the second expression for y_h:

$$\omega t_j - \theta = \frac{\pi}{2} = \pi j \qquad \text{for} \quad j = 0, 1, 2, 3, \ldots$$

so that

$$t_j = \frac{1}{\omega}\left[\theta + \left(j + \frac{1}{2}\right)\pi\right].$$

These are the possible candidates for stopping times T. Also, we note that for each $j = 0, 1, 2, 3, \ldots$

$$\cos(\omega t_j - \theta) = 0 \qquad \text{and} \qquad \sin(\omega t_j - \theta) = (-1)^j.$$

Finally, we note the alternative form of y_h':

$$y_h'(t) = \sqrt{a^2+b^2}e^{-pt/2}\left[-\frac{p}{2}\cos(\omega t - \theta) - \omega\sin(\omega t - \theta)\right].$$

From this we can see the incoming momentum at $T = t_j$, namely

$$my_h'(t_j) = m\omega(-1)^{j+1}\sqrt{a^2+b^2}e^{-pT/2}. \tag{1}$$

We will use this later.

Now, the impulse at time $T = t_j$ is modeled in Laplace transform space by

$$(ms^2 + \alpha s + k)Y - my_0 s - my_0' - \alpha y_0 = \mu e^{-sT}.$$

That is,

$$(s^2 + ps + q)Y = y_0 s + y_0' + py_0 + \frac{\mu}{m}e^{-sT}.$$

Therefore,

$$\begin{aligned}
Y &= y_0 \frac{s+p/2}{(s+p/2)^2 + q - p^2/4} + \frac{y_0' + py_0/2}{(s+p/2)^2+\omega^2} + \frac{\mu}{m}\frac{e^{-sT}}{(s+p/2)^2+\omega^2} \\
&= y_0 \mathcal{L}\left[e^{-pt/2}\cos\omega t\right] + \frac{y_0' + py_0/2}{\omega}\mathcal{L}\left[e^{-pt/2}\sin\omega t\right] \\
&\qquad + \frac{\mu}{m\omega}\mathcal{L}\left[H(t-T)e^{-p(t-T)/2}\sin\omega(t-T)\right].
\end{aligned}$$

This means

$$\begin{aligned}
y(t) &= e^{-pt/2}\left[a\cos\omega t + b\sin\omega t + \frac{\mu}{m\omega}H(t-T)e^{pT/2}\sin[\omega t - \theta - (j+1/2)\pi]\right] \\
&= e^{-pt/2}\left[\sqrt{a^2+b^2} - (-1)^j\frac{\mu}{m\omega}H(t-T)e^{pT/2}\right]\cos(\omega t - \theta).
\end{aligned}$$

Note that we have plugged in the explicit value $T = t_j$ into the trigonometric function $\sin\omega(t-T)$ and then expanded using the sine addition formula and the values of $\sin(\omega t_j - \theta)$ and $\cos(\omega t_j - \theta)$ given above. From the last expression for y it is clear that in order to have $y(t) \equiv 0$ for $t \geq T$, we need

$$\mu = m\omega(-1)^j\sqrt{a^2+b^2}e^{pT/2}.$$

This essentially completes the calculation. We may now look back at the momentum (1) and see that μ is precisely the negative of the value we obtained.

We now apply the calculation to a specific case with $L[y] = y'' + 2y' + 9y$, $y(0) = 3$, $y'(0) = 0$. In this case, $p = \alpha = 2$ and $q = k = 9$, $\lambda = -1 \pm 2i\sqrt{2}$, and

$$\begin{aligned}
y_h &= e^{-t}\left(3\cos 2\sqrt{2}\,t + \frac{3}{2\sqrt{2}}\sin 2\sqrt{2}\,t\right) \\
&= \frac{9}{2\sqrt{2}}e^{-t}\cos(2\sqrt{2}\,t - \theta),
\end{aligned}$$

where

$$\cos\theta = \frac{2\sqrt{2}}{3} \qquad \text{and} \qquad \sin\theta = \frac{1}{3}.$$

That is, $\theta = \sin^{-1}(1/3)$. Calculating y_h', we see

$$t_j = \frac{1}{2\sqrt{2}}\left[\theta + \left(j + \frac{1}{2}\right)\pi\right], \quad j = 0, 1, 2, 3, \ldots$$

Since the mass $m = 1$, the incoming momentum is $y_h'(t_j) = 9(-1)^{j+1}e^{-t_j}$, and we take

$$\mu = 9(-1)^j e^{-tj}.$$

The resulting motion is described by

$$\frac{9\sqrt{2}}{4}e^{-t}\left[1 - H(t - t_j)\right]\cos\left[2\sqrt{2}\,t - \sin^{-1}(1/3)\right].$$

Exercise 1 *Use numerical software like Matlab or Mathematica to plot the motions* y *for the first few stopping times* t_0, t_1, t_2, $\ldots$.

Exercise 2 *Solve the undamped problem from the beginning. Use an operator* $L[y] = y'' + ky$ *where* k *is a squared integer and take* $y(0) = y_0$ *with* $y'(0) = 0$.

Exercise 3 *Go back and derive the possible stopping times directly from the solution you get from the Laplace transform method using an arbitrary stopping time* T.