

# Calculus Review for ODE

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## 1 The Fundamental Theorem of Calculus

We begin with a mathematical (calculus) problem:

*Find a function  $y = y(x)$  whose derivative is given (prescribed) by another (continuous) function  $f = f(x)$ .*

This problem asks us to solve the equation  $y'(x) = f(x)$  for the function  $y$ . The Fundamental Theorem of Calculus tells us how:

*If  $f = f(x)$  is any continuous function,  $x_0$  is in the domain of definition of  $f$ , and  $y_0$  is any constant, then the function*

$$y(x) = y_0 + \int_{x_0}^x f(t) dt$$

*has a well defined derivative, and that derivative is given by the formula*

$$y'(x) = f(x).$$

The answer given by the Fundamental Theorem may be somewhat unexpected; there is not just one such function but a family of solutions. If we know specific choices for the starting point  $x_0$  and the starting value  $y_0$ , then we get a unique solution.

Apply the Fundamental Theorem of Calculus to the following problems:

1. Find the function  $y = y(x)$  whose derivative is given by  $f(x) = x^2 + 3$  and which satisfies  $y(2) = 1$ .
2. Find the function  $y = y(x)$  whose derivative is given by  $f(x) = \sin^2 x$  and which satisfies  $y(2) = 1$ .
3. Find the function  $y = y(x)$  whose derivative is given by  $f(x) = x \sin^2 x$  and which satisfies  $y(0) = 1$ .
4. Find the function  $y = y(x)$  whose derivative is given by  $f(x) = \sin x^2$  which satisfies  $y(0) = 1$ .

Here is the same kind of problem reworded in a way that may be more familiar to you:

- Find the velocity  $v = v(t)$  of an object whose acceleration is given by  $a(t) = -9.8$  (meters/second)/second and whose velocity at time  $t = 2$  is 5 meters/second. Describe a physical system which might be described by this problem.

Here are some more:

- A stone is dropped from a height of 98 meters. In how many seconds does it hit the ground?
- An explosion causes debris to rise vertically with an initial velocity of 72 feet per second. In how many seconds does it attain a maximum height? What is the maximum height?
- A stone was thrown up at a speed of 2 meters per second. After 2.5 seconds, the stone was caught by the person who threw it. What was the maximum height, and who threw the stone?

Finally, here are a couple “thinking” problems. (Everybody needs to be a thinker in this course.)

- In the Fundamental Theorem why would it be wrong to say

$$y(x) = y_0 + \int_{x_0}^x f(x) dx ?$$

- The version of the Fundamental Theorem of Calculus with which you are probably familiar addresses how to compute an integral:

*If  $F = F(x)$  is a differentiable function whose derivative is  $f(x)$ , then*

$$\int_a^b f(x) dx = F(b) - F(a).$$

Explain why these two versions really say the same thing. (Make sure you explain the correspondence between  $f, F, a, x$ , and  $b$  in this statement and  $y, f, x_0, x, y_0$ , and  $t$  in the one given at the beginning of this worksheet.)

## 2 Notes

- The mathematical concepts on this worksheet should be familiar to you. If they are not, you may wish to review sections 5.1 and 5.2 of *Calculus* by Salas, Hille, and Etgen. Page 149 may also be of interest. If you want to read ahead, you can also review sections 18.1 and 7.6.
- Problem 4.

<http://www.math.unt.edu/integration-bee/>

Scroll down to the section “The Awful Truth.”

- Problem 8.

<http://hypertextbook.com/facts/2004/MichaelRobbins.shtml>