Math 2552, Exam 2 (practice) Name and section: ________________

This exam covers chapter 3, chapter 4, and parts of chapter 6 and part of chapter 8 of Brannon and Boyce. The exam covers some material on numerics, first order systems, and second order linear ODE. A more precise outline of topics you should know is the following:

1. numerics
   (a) Euler's method
   (b) improved Euler method
2. first order systems
   (a) basic defintions
   (b) linear -vs- nonlinear
   (c) linear existence and uniqueness
   (d) homogeneous linear systems with constant coefficients
      i. eigenvalue/eigenvector (straight line) solutions
      ii. basis of real eigenvectors
      iii. complex eigenvectors
      iv. one dimensional eigenspace
      v. general solution (in each case)
      vi. phase diagram (in each case)
      vii. stability classification of equilibria (in each case)
      viii. names (saddle, stable sink, unstable source, stable spiral, etc.)
      ix. asymptotic stability
   (e) nonlinear systems
      i. nonlinear existence and uniqueness theorem
      ii. autonomous case
         A. equilibrium points
         B. linearization (*not on this exam)
         C. phase plane diagram techniques (*not on this exam)
3. linear second order ODE
   (a) homogeneous equations with constant coefficients
   (b) equivalence with first order systems
   (c) finding particular solutions with forcing
   (d) general solutions (particular plus homogeneous)
4. modeling
   (a) populations systems (logistic, competition, etc.) (*not on this exam)
   (b) elementary oscillators

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1. (17 points) (3.2.8) Express the following system using vector/matrix notation.

$$\begin{cases} x' = 3x - 4y \\ y' = x + 3y. \end{cases}$$

Solution: This system can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3x - 4y \\ x + 3y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

or simply

$$\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & 3 \end{pmatrix} \mathbf{x}.$$

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2. (17 points) (3.2.21) Give a first order system which is equivalent to the single ODE

$$y''' + 3y'' - y = 0.$$

Solution: Letting $x_1 = y$ be one unknown in our system, we define two more unknowns by $x_2 = x_1'$ and $x_3 = x_2'$. The equivalent first order system is then

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -x_1 - 3x_3. \end{cases}$$

or simply

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -3 \end{pmatrix} \mathbf{x}.$$

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3. (17 points) (3.3.11) Solve the linear system of ODEs

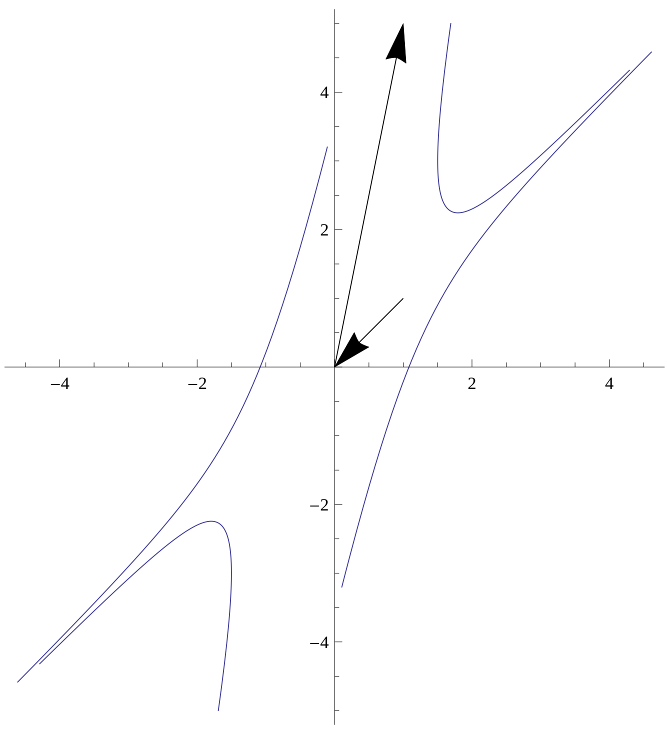
$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x},$$

and plot the phase diagram.

Solution: The characteristic equation is $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$. For the eigenvalue $\lambda_1 = -1$, we have an eigenvectore $\mathbf{v} = (v_1, v_2)^T$ satisfying $-v_1 + v_2 = 0$. One such vector is $\mathbf{v} = (1, 1)^T$. Similarly, for the eigenvalue $\lambda_2 = 3$ we find an eigenvector $\mathbf{w} = (1, 5)^T$. Therefore, the general solution is

$$\mathbf{x}(t) = ae^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + be^{3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

This is a saddle (unstable):



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4. (rabbits and foxes) Consider the system

$$\begin{cases} r' = 3r(2 - r - 4f) \\ f' = r - 2f \end{cases}$$

for two populations r and f which change over time.

(a) (8 points) Find any equilibrium populations for the system.

(b) (9 points) Find the Euler approximation for $r(1)$ and $w(1)$ if $(r, w)^T$ is the solution starting with $r(0) = 5$ and $w(0) = 3$, and the stepsize is $1/2$.

Solution:

(a) We wish to solve the algebraic system $3r(2-r-4f) = 0$ and $r - 2f = 0$. From the first equation $r = 0$ or $r + 4f = 2$. In the first case, the second equation implies $f = 0$, so one equilibrium point is the origin where both populations are zero for all time. In the second case, we find $f = 1/3$ and $r = 2/3$. Thus, the two equilibrium points are

$$\begin{pmatrix} r_* \\ f_* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} r_* \\ f_* \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}.$$

(b) The approximation for $(r(1/2), w(1/2))^T$ is obtained as follows:

$$\begin{pmatrix} r_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \left(\frac{1}{2}\right) \begin{pmatrix} 15(2-5-12) \\ 5-6 \end{pmatrix} = -215/2\,(\ 5/2\ ).$$

The required approximation for $(r(1), w(1))^T$ is therefore

$$\begin{pmatrix} r_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} -215/2 \\ 1/2 \end{pmatrix} + \left(\frac{1}{2}\right) \begin{pmatrix} -(645/2)(215/2) \\ -217/2 \end{pmatrix} = \begin{pmatrix} -139/535/8 \\ -215/4 \end{pmatrix}.$$

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5. (17 points) Solve the initial value problem

$$\begin{cases} y'' + 4y = \cos 3t \\ y(0) = 0 = y'(0) \end{cases}$$

determine the *period* of the beats.

Solution:

$$y_h = a\cos 2t + b\sin 2t.$$

$$y_p = A\cos 3t + B\sin 3t.$$

$$L[y_p] = -5A\cos 3t - 5B\sin 3t = \cos 3t.$$

Thus,

$$y(t) = a\cos 2t + b\sin 2t - \frac{1}{5}\cos 3t$$

for some constants a and b.

$$y(0) = a - 1/5 = 0 \quad \text{and} \quad y'(0) = 2b = 0.$$

Therefore,

$$y(t) = \frac{1}{5}(\cos 2t - \cos 3t).$$

Setting

$$\theta = \frac{2t + 3t}{2} \quad \text{and} \quad \phi = \frac{2t - 3t}{2}$$

and using the cosine addition formula, we find

$$y(t) = \frac{2}{5}\sin\left(\frac{5t}{2}\right)\sin\left(\frac{t}{2}\right).$$

The beats are determined by the sine function of lower frequency. That would be frequency 1/2. And the period is therefore

$$\frac{2\pi}{1/2} = 4\pi.$$