Math 2403, Exam 2 (practice) Name and section: ________________

This exam covers chapter 3, chapter 7 (sections 1-4), and part of chapter 4 of Brannon and Boyce. The exam covers some material on first order systems and second order linear ODE. A more precise outline of topics you should know is the following:

1. first order systems
   (a) basic defintions
   (b) linear -vs- nonlinear
   (c) linear existence and uniqueness
   (d) homogeneous linear systems with constant coefficients
      i. eigenvalue/eigenvector (straight line) solutions
      ii. basis of real eigenvectors
      iii. complex eigenvectors
      iv. one dimensional eigenspace
      v. general solution (in each case)
      vi. phase diagram (in each case)
      vii. stability classification of equilibria (in each case)
      viii. names (saddle, stable sink, unstable source, stable spiral, etc.)
      ix. asymptotic stability
   (e) nonlinear systems
      i. nonlinear existence and uniqueness theorem
      ii. autonomous case
         A. equilibrium points
         B. linearization
         C. phase plane diagram techniques
2. linear second order ODE
   (a) homogeneous equations with constant coefficients
   (b) equivalence with first order systems
   (c) finding particular solutions with forcing
   (d) general solutions (particular plus homogeneous)
3. modeling
   (a) populations systems (logistic, competition, etc.)
   (b) elementary oscillators

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1. (17 points) (3.2.8) Express the following system using vector/matrix notation.

$$\begin{cases} x' = 3x - 4y \\ y' = x + 3y. \end{cases}$$

Solution: This system can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3x - 4y \\ x + 3y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

or simply

$$\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & 3 \end{pmatrix} \mathbf{x}.$$

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2. (17 points) (3.2.21) Give a first order system which is equivalent to the single ODE

$$y''' + 3y'' - y = 0.$$

Solution: Letting $x_1 = y$ be one unknown in our system, we define two more unknowns by $x_2 = x_1'$ and $x_3 = x_2'$. The equivalent first order system is then

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -x_1 - 3x_3. \end{cases}$$

or simply

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -3 \end{pmatrix} \mathbf{x}.$$

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3. (17 points) (3.3.11) Solve the linear system of ODEs

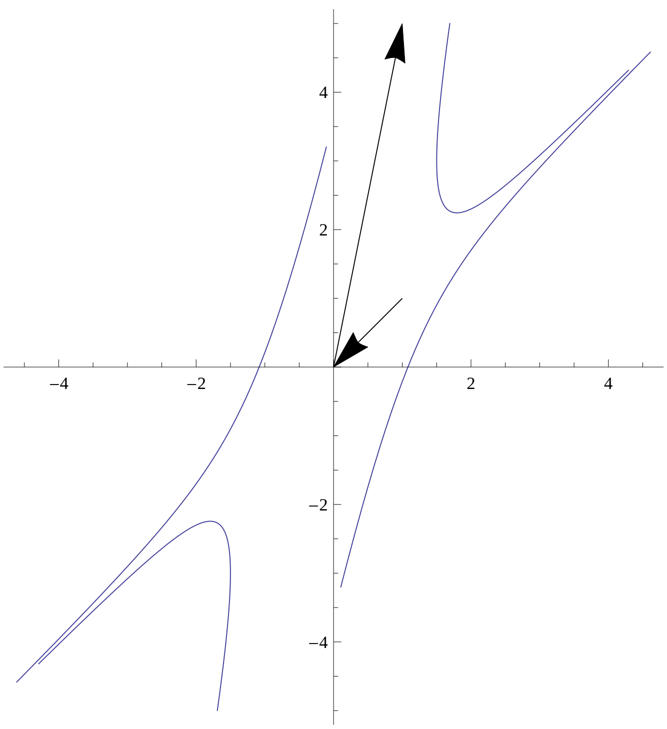
$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x},$$

and plot the phase diagram.

Solution: The characteristic equation is $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$. For the eigenvalue $\lambda_1 = -1$, we have an eigenvectore $\mathbf{v} = (v_1, v_2)^T$ satisfying $-v_1 + v_2 = 0$. One such vector is $\mathbf{v} = (1, 1)^T$. Similarly, for the eigenvalue $\lambda_2 = 3$ we find an eigenvector $\mathbf{w} = (1, 5)^T$. Therefore, the general solution is

$$\mathbf{x}(t) = ae^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + be^{3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

This is a saddle (unstable):



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4. (3.5.6) Consider the linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x},$$

(a) (6 points) Draw the phase diagram.

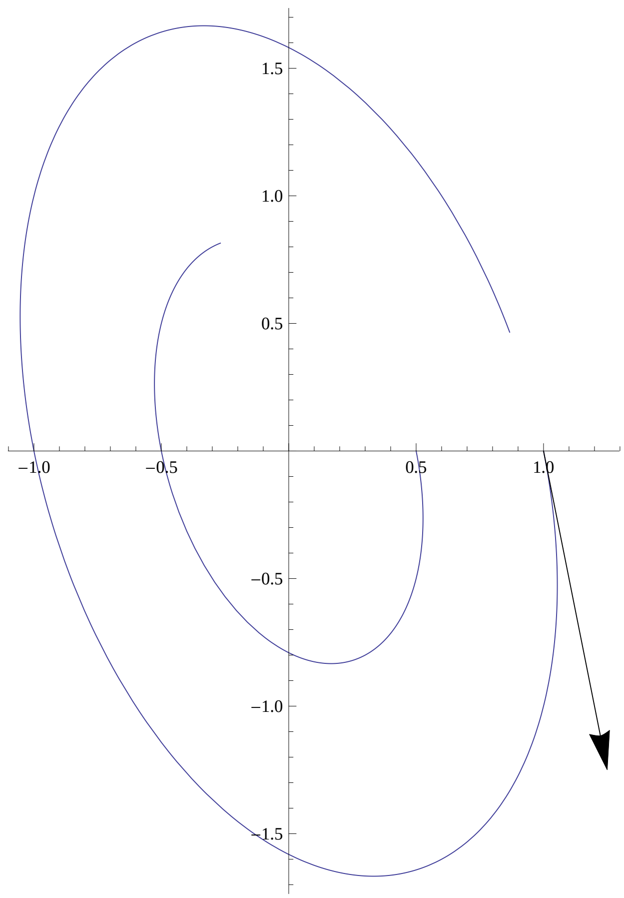
(b) (6 points) If a solution satisfies $\mathbf{x}(0) = (3, 2)^T$, then what can you say about

$$\lim_{t\to\infty} \sqrt{x_1(t)^2 + x_2(t)^2}.$$

(c) (5 points) Classify the equilibrium point.

Solution:

(a) The characteristic equation is $\lambda^2 + 9 = 0$. Thus, the eigenvalues are $\pm 3i$ and are purely imaginary. Inspection of the direction field indicates the the trajectories/orbits are (noncircular) ellipses with major axis passing through the second and fourth quadrants and are traversed by solutions in the clockwise direction:



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(b) The solution passing through this point will be a clockwise parameterization of an ellipse passing through $(3, 2)^T$. As a consequence, the quantity in question, which is the distance from $\mathbf{x}(t)$ to the origin will oscillate periodically between some positive minimum and maximum values. Therefore, *the limit does not exist*.

(c) The equilibrium point at the origin is called an elliptic periodic center.

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5. (rabbits and foxes) Consider the system

$$\begin{cases} r' = 3r(2 - r - 4f) \\ f' = r - 2f \end{cases}$$

for two populations r and f which change over time.

(a) (8 points) Find any equilibrium populations for the system.

(b) (9 points) Linearize the system at each equilibrium point, and classify the local behavior there if possible. If linearization does not lead to a definitive characterization, explain why.

Solution:

(a) We wish to solve the system $3r(2 - r - 4f) = 0$ and $r - 2f = 0$. From the first equation $r = 0$ or $r + 4f = 2$. In the first case, the second equation implies $f = 0$, so one equilibrium point is the origin where both populations are zero for all time. In the second case, we find $f = 1/3$ and $r = 2/3$. Thus, the two equilibrium points are

$$\begin{pmatrix} r_* \\ f_* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} r_* \\ f_* \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}.$$

(b) Denoting the vector field defining the ODE by

$$F\begin{pmatrix} r \\ f \end{pmatrix} = \begin{pmatrix} 3r(2 - r - 4f) \\ r - 2f \end{pmatrix} = \begin{pmatrix} 3(2r - r^2 - 4rf) \\ r - 2f \end{pmatrix},$$

we find

$$DF\begin{pmatrix} r \\ f \end{pmatrix} = \begin{pmatrix} 3(2 - 2r - 4f) & -12r \\ 1 & -2 \end{pmatrix}.$$

Thus, for the equilibrium at the origin, the linearized system is

$$\mathbf{y}' = \begin{pmatrix} 6 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{y}.$$

The eigenvalues for the matrix $DF(0,0)^T$ are 6 and -2. Since these have opposite signs, there is a saddle at the origin.

For the second equilibrium point, the linearized system is

$$\mathbf{y}' = \begin{pmatrix} -2 & -8 \\ 1 & -2 \end{pmatrix} \mathbf{y}.$$

The characteristic equation in this case is $\lambda^2 + 4\lambda + 12 = 0$. The roots are

$$\lambda = \frac{-4 \pm \sqrt{16 - 48}}{2} = -2 \pm 2\sqrt{3}i.$$

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This means the second equilibrium is a stable inward spiral. Checking the vector field at a point $(2/3, 1/3 + \epsilon)$ where ϵ is a small positive number, we find the second component of F is

$$r - 2f = -2\epsilon < 0.$$

This means we have a clockwise spiral at $(2/3, 1/3)^T$.

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6. (17 points) (4.3.8) Find the genearal solution of the ODE

$$2y'' - 3y' + y = 0.$$

Solution: This is a second order linear ODE with constant coefficients, and it is homogeneous. Therefore, we look for solutions of the form

$$y(t) = e^{\alpha t}.$$

Plugging in, we obtain the characteristic equation

$$2\alpha^2 - 3\alpha + 1 = (2\alpha - 1)(\alpha - 1) = 0.$$

Thus, the general solution is

$$y(t) = c_1 e^{t/2} + c_2 e^t$$

where c_1 and c_2 are arbitrary constants.