Math 2403, Exam 1 (practice) Name and section: ____________________

This exam covers chapters 1 and 2 of Brannon and Boyce. Everything on the exam pertains to *a single first order equation.* It can also be summarized under four headings:

1. single first order equations
2. modeling
3. numerical methods
4. existence/uniqueness

As an expanded study outline is as follows:

1. single first order equations
   (a) FTC
   (b) first order linear
   (c) separable
   (d) exact
   (e) autonomous
2. existence/uniqueness
   (a) linear
   (b) nonlinear
3. modeling
   (a) mixing/flow problems
   (b) exponential growth/decay (interest/population/radiation)
   (c) Newton's law of cooling
4. numerical methods
   (a) slope fields
   (b) Euler method
   (c) improved Euler method
   (d) Runge-Kutte method
   (e) phase diagram (autonomous)

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1. (17 points) (1.2.9) Solve the following ODE for $y = y(t)$:

$$2y' + y = 3t.$$

Solution: This is a linear first order equation $y' + p(t)y = 3t/2$. Using the integrating factor $\mu = \exp(\int^t p) = \exp(t/2)$, we write

$$\left(ye^{t/2}\right)' = (3t/2)e^{t/2}.$$

Integrating both sides from t_0 to t, we get

$$\begin{aligned} ye^{t/2} - y(t_0)e^{t_0/2} &= \frac{3}{2}\int_{t_0}^{t} \tau e^{\tau/2}\, d\tau \\ &= \frac{3}{2}\left[\left(2\tau e^{\tau/2}\right)\Big|_{t_0}^{t} - \int_{t_0}^{t} 2e^{\tau/2}\, d\tau\right] \\ &= 3te^{t/2} - 3t_0e^{t_0/2} - 6e^{t/2} + 6e^{t_0/2}. \end{aligned}$$

Thus,

$$y(t) = 3t - 6 + (y_0 - 3t_0 + 6)e^{t_0/2}\, e^{-t/2},$$

or alternatively $y(t) = 3t - 6 + ce^{-t/2}$ where c is a constant.

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2. (17 points) (2.1.10) Solve the initial value problem (IVP) for $y = y(x)$:

$$\begin{cases} y' = (1-2x)/y \\ y(1) = -2. \end{cases}$$

Solution: This is a separable equation $yy' = 1 - 2x$. Integrating both sides from 1 to x, we find

$$\frac{1}{2}y^2 - 2 = x - 1 - x^2 + 1 = x - x^2.$$

Thus, $y = -\sqrt{2(2-x)(x+1)}$ for $-1 < x < 2$.

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3. (17 points) (2.2.4) A tank contains 200 gallons of water with 100 lbs of salt in solution. Solution with 1 lb of salt per gallon is added at 3 gal per minute. Assume instantaneous mixing and an outflow of the resulting mix at 2 gal per minute. Find the concentration of salt as a function of time and the limiting concentration as time tends to ∞.

Solution: Let $S = S(t)$ be the amount of salt in the tank at time t. Then

$$\begin{cases} S' = 3 - \frac{S}{200+t} \cdot 2 \\ \\ S(0) = 100. \end{cases}$$

The ODE is linear with integrating factor

$$\mu = e^{\int^t \frac{2}{200+t}} = e^{\ln(200+t)^2} = (200+t)^2.$$

Thus, we get

$$[(200+t)^2 S]' = 3(200+t)^2.$$

Integrating both sides from 0 to t, we get

$$(200+t)^2 S - 200^2 \cdot 100 = (200+t)^3 - 200^3.$$

Thus,

$$\begin{aligned} S &= \frac{1}{(200+t)^2}\left[(200+t)^3 - 100 \cdot 200^2\right] \\ &= 200 + t - \frac{100(200)^2}{(200+t)^2}. \end{aligned}$$

To get the concentration $C = C(t)$ we divide the amount of salt by the total volume of solution, $200 + t$:

$$C(t) = 1 - \frac{100(200)^2}{(200+t)^3}.$$

We then have

$$\lim_{t \nearrow \infty} C(t) = 1,$$

as one would expect since the concentration of the inflow is 1 lb per gallon.

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4. Discuss the implications of the existence and uniqueness theorems for the following equation and IVP. (In each case, $y = y(t)$.)

   (a) (8 points) (2.3.3)

$$\begin{cases} y' + y \tan t = \sin t \\ y(\pi) = 0. \end{cases}$$

   (b) (9 points) (2.3.12)

$$y' = \frac{y \cot t}{1 + y}.$$

Solution:

(a) This equation is linear, and it will have a unique solution on any interval for which the coefficients are C^1. The only singularities are those of $\tan t$, and the maximum interval containing $t = \pi$ on which $\tan t$ is regular is the interval $(\pi/2, 3\pi/2)$. Therefore, this is the interval on which the linear existence and uniqueness theorem guarantees the existence of a unique solution.

(b) This is a nonlinear equation to which the nonlinear existence and uniqueness theorem applies. This theorem relies on the regularity of the function

$$f(y, t) = \frac{y \cot t}{1 + y}$$

jointly in y and t. The singulariities of this function lie along the union of the horizontal line $y = -1$ and the vertical lines $t = k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$. For each point (t_0, y_0) in the complement of these lines, there is some time interval $(t_0 - \epsilon, t_0 + \epsilon)$ on which the ODE has a unique solution. This is all we can say from the nonlinear existence/uniqueness theorem.

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5. (17 points) (2.4.2) Find and classify the equilibrium points of the ODE $y' = 3y + y^2$.

Solution: Let $f(y) = 3y + y^2 = y(3 + y)$. The equilibrium point at $y_* = 0$ has $f'(0) = 3 > 0$. Therefore, this is an unstable source.

The equilibrium point at $y_* = -3$ has $f'(-3) = -3 < 0$. Therefore, this is a stable sink.

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6. (2.6.3) Consider the IVP

$$\begin{cases} y' = 2y - 3t \\ y(0) = 1. \end{cases}$$

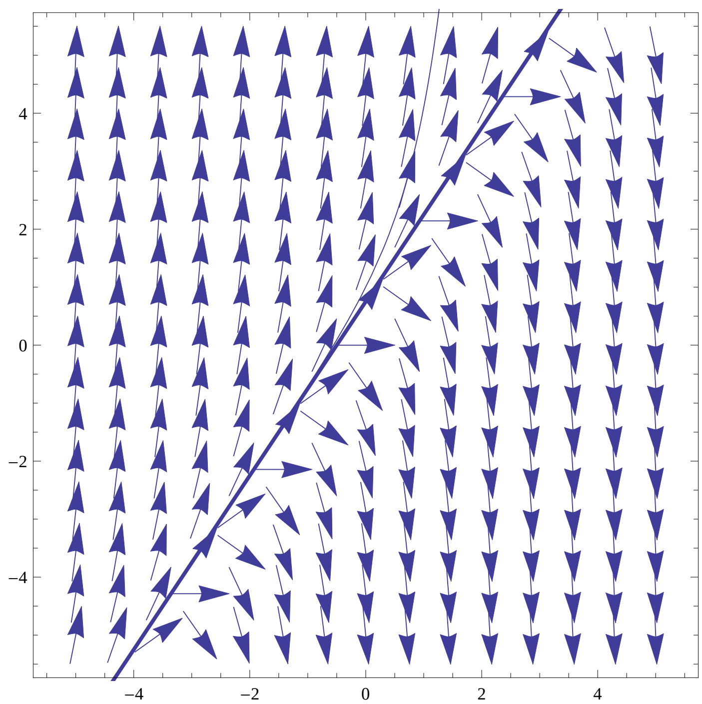
(a) (6 points) Sketch the slope field for the ODE and indicate the solution graphically.

(b) (6 points) Find the Euler approximation for $y(0.2)$ with step size $h = 0.1$.

(c) (5 points) Find the improved Euler approximation for $y(0.1)$ with step size $h = 0.1$.

Solution:

(a)



Note that I've drawn in bold the line $y = 3(1 + 2t)/4$ which is a solution of the ODE though not the solution of the IVP which diverges (with greater rate of growth) from the line.

(b) $y_1 = 1 + 2(0.1) = 1.2.$

$$\begin{aligned} y_2 &= 1.2 + [2(1.2) - 3(0.1)](0.1) \\ &= 1.2 + [2.4 - 0.3](0.1) \\ &= 1.2 + 0.21 \\ &= 1.41. \end{aligned}$$

This last number gives the desired approximation $y(0.2) \approx 1.41$.

(c) We can determine the two relevant slopes from the previous calculation: $m_0 = 2$ and $m_1 = 2.4 - 0.3 = 2.1$. Thus, the average is

$$m = \frac{4.1}{2} = 2.05$$

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and the improved Euler approximation (first step) is

$$y(0.1) \approx \tilde{y}_1 = 1 + 2.05(0.1) = 1.205.$$