Math 2552, Exam 1 form s Name and section: ____________________

1. (Chapter 2: classification and solution of an ODE)

   Here is an ordinary differential equation for a function $y = y(x)$:

$$y' = \frac{(x-1)y^5}{x^2(2y^3 - y)}.$$

   (a) (10 points) Classify the ODE. (You should specify at least three things.)

   (b) (10 points) Solve the ODE.

Solution:

(a) separable, first order, nonlinear, nonautonomous, exact

(three points each for correct answers and one free point)

(b)

$$(2y^{-2} - y^{-4})y' = x^{-1} - x^{-2}.$$

$$-\frac{2}{y} + \frac{1}{3y^3} = \ln|x| + \frac{1}{x} + c$$

where c is a constant and y is given implicitly.

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2. (Chapter 2: classification and solution of an IVP)

   Here is an initial value problem for a function $y = y(t)$:

$$y' = 2ty + 3t^2e^{t^2}, \qquad y(0) = 5.$$

   (a) (10 points) Classify the ODE in the IVP. (You should specify at least three things.)

   (b) (10 points) Solve the IVP.

Solution:

(a) first order, linear, nonhomogeneous, nonautonomous

(three points each for correct answers and one free point)

(b)

$$e^{-t^2}y = 5 + t^3.$$

Therefore, we have

$$y = (5 + t^3)e^{t^2}.$$

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3. (Section 2.3: applications)

(a) (10 points) A body cools in a 70 degree room according to Newton's law of cooling. At noon the body is 80 degrees and at 1PM the body is 75 degrees. When was the body 98.6 degrees?

(b) (10 points) Three tanks, numbered 1, 2 and 3, contain well-mixed ethanol-water mixture. Mixture with concentration γ_j of ethanol flows into tank j at a rate r_j gallons per minute, $j = 1, 2, 3$. Well mixed liquid flows from tank i to tank j for every $i \neq j$ at a rate e_{ij} and flows out of the system from tank j at a rate f_j. If the initial concentration of liquid in tank j is Γ_j and the initial volume is W_j, write down the conditions on the rates that result in an autonomous system for the amounts of ethanol in each tank.

(c) (10 points) If the system for the amounts of ethanol in each tank from the previous part only involves two tanks (tank 1 and tank 2) and is autonomous, and $r_1 = f_1 = r$ and $W_1 = W_2 = W$, find the equilibrium point(s) of the system.

Solution:

(a) $T' = c(70 - T)$, $T(0) = 98.6$. $e^{ct}T = 98.6 + 70e^{ct} - 70$.

$$\begin{aligned} T(t_1) &= 28.6e^{-ct_1} + 70 = 80 \qquad \text{(noon)}. \\ T(t_1+1) &= 28.6e^{-ct_1-c} + 70 = 75. \end{aligned}$$

$e^c = 2$, so $c = \ln 2$, and $28.6e^{-t_1 \ln 2} = 10$. Thus, $t_1 \ln 2 = \ln 2.86$, and $t_1 = \ln 2.86/\ln 2$, so the time was about $\ln 2.86/\ln 2 \approx 1.52$ hours before noon, or at about 10:30 AM. Note: You won't be able to use a calculator on the exam, so your answer will/should be just "$\ln 2.86/\ln 2$ hours before noon."

(b)

$$\begin{aligned} r_1 - e_{12} - e_{13} + e_{21} + e_{31} - f_1 &= 0 \\ r_2 - e_{21} - e_{23} + e_{12} + e_{32} - f_2 &= 0 \\ r_3 - e_{31} - e_{32} + e_{13} + e_{23} - f_3 &= 0. \end{aligned}$$

(c) The system simplifies to

$$\begin{cases} A_1' = \gamma_1 r_1 - (e_{12} + r_1)A_1/W + e_{12}A_2/W, & A_1(0) = \Gamma_1 W \\ A_2' = \gamma_2 r_2 + e_{12}A_1/W - (e_{12} - r_2)A_2/W, & A_2(0) = \Gamma_2 W. \end{cases}$$

So we are interested in solving

$$\begin{cases} (e_{12} + r_1)A_1^* - e_{12}A_2^* = \gamma_1 W r_1 \\ -e_{12}A_1^* + (e_{12} + r_2)A_2^* = \gamma_2 W r_2. \end{cases}$$

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By Cramer's rule:

$$A_1^* = W\frac{\gamma_1 r_1(e_{12}+r_2)+\gamma_2 r_2 e_{12}}{(r_1+r_2)e_{12}+r_1 r_2} = W\frac{(\gamma_1 r_1+\gamma_2 r_2)e_{12}+\gamma_1 r_1 r_2}{(r_1+r_2)e_{12}+r_1 r_2}$$

and

$$A_2^* = W\frac{\gamma_1 r_1 e_{12}+\gamma_2 r_2(e_{12}+r_1)}{(r_1+r_2)e_{12}+r_1 r_2} = W\frac{(\gamma_1 r_1+\gamma_2 r_2)e_{12}+\gamma_2 r_1 r_2)}{(r_1+r_2)e_{12}+r_1 r_2}.$$

Incidentally, an interesting case is when one has also $r_1 = r_2 = r$. Then the equilibrium levels of ethanol are

$$A_1^* = W\frac{(\gamma_1+\gamma_2)e_{12}+\gamma_1 r}{2e_{12}+r}$$

and

$$A_2^* = W\frac{(\gamma_1+\gamma_2)e_{12}+\gamma_2 r}{2e_{12}+r}.$$

Notice that if $\gamma_1 > \gamma_2$, then

$$A_1^* - A_2^* = W\frac{(\gamma_1-\gamma_2)r}{2e_{12}+r} > 0.$$

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4. (Autonomous equations and exact equations)

(a) (10 points) Draw the phase diagram for the Gompertz equation

$$P' = 3P\ln(6/P).$$

(b) (10 points) Determine if the equation is exact. If it is exact, solve the equation.

$$\frac{x^3}{y^2} + \frac{3}{y} + \frac{3}{4}\left(\frac{x}{y^2} + 4y\right)y' = 0.$$

Solution:

(a) 

Notice that zero is not an equilibrium point here, nor is the interval $(-\infty, 0]$ really any part of the phase space/line. The phase line is the open interval $(0, \infty)$.

(b)

$$\frac{\partial}{\partial x}\frac{3}{4}\left(\frac{x}{y^2} + 4y\right) = \frac{3}{4}\left(\frac{1}{y^2}\right),$$

and

$$\frac{\partial}{\partial}\frac{x^3}{y^2} + \frac{3}{y} = -\frac{2x^3}{y^3} - \frac{3}{y^2}.$$

Since

$$\frac{3}{4}\left(\frac{1}{y^2}\right) \neq -\frac{2x^3}{y^3} - \frac{3}{y^2},$$

the equation is not exact.

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5. (Chapter 3: A system of ODEs) Here is a system of ODES.

$$\begin{cases} x' = -13x + 6y + 112 \\ y' = 2x - 2y. \end{cases}$$

(a) (10 points) Solve the system.

(b) (10 points) Draw the phase diagram.

Solution:

(a) First we find the equilibrium point(s). We want to find x^* and y^* such that

$$\begin{cases} -13x^* + 6y^* + 112 = 0 \\ 2x^* - 2y^* = 0. \end{cases}$$

Inverting the matrix

$$A = \begin{pmatrix} -13 & 6 \\ 2 & -2 \end{pmatrix}$$

we get

$$\begin{pmatrix} -13 & 6 \\ 2 & -2 \end{pmatrix}^{-1} = \frac{1}{14}\begin{pmatrix} -2 & -6 \\ -2 & -13 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \frac{1}{14}\begin{pmatrix} -2 & -6 \\ -2 & -13 \end{pmatrix}\begin{pmatrix} -112 \\ 0 \end{pmatrix} = \begin{pmatrix} 16 \\ 16 \end{pmatrix}.$$

Next, we consider the associated homogeneous system $\mathbf{x}' = A\mathbf{x}$. The characteristic equation is

$$(-13 - \lambda)(-2 - \lambda) - 12 = \lambda^2 + 15\lambda + 14 = (\lambda + 14)(\lambda + 1) = 0.$$

(It's a stable sink.) $\lambda = -14$:

$$\begin{pmatrix} 1 & 6 \\ 2 & 12 \end{pmatrix} \quad \text{has null space spanned by} \quad \begin{pmatrix} -6 \\ 1 \end{pmatrix}.$$

$\lambda = -1$:

$$\begin{pmatrix} -12 & 6 \\ 2 & -1 \end{pmatrix} \quad \text{has null space spanned by} \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Therefore, the solution of the homogeneous system is

$$\mathbf{x} = c_1 e^{-14t}\begin{pmatrix} -6 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Since

$$A\mathbf{x}^* = \begin{pmatrix} -112 \\ 0 \end{pmatrix}$$

we know

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^* \\ y^* \end{pmatrix}\right].$$

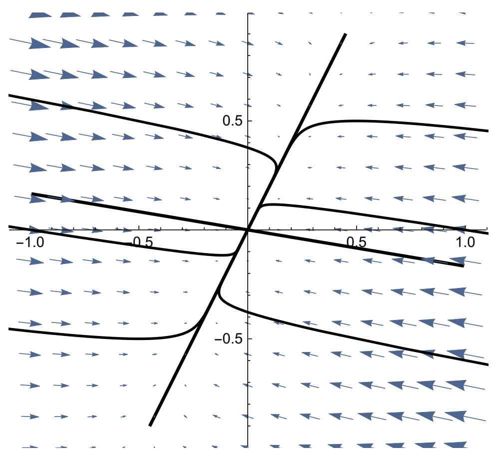
This means

$$\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^* \\ y^* \end{pmatrix}$$

is a solution of the associated homogeneous system, so the general solution of the original problem is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1e^{-14t}\begin{pmatrix} -6 \\ 1 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} x^* \\ y^* \end{pmatrix} = c_1e^{-14t}\begin{pmatrix} -6 \\ 1 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 16 \\ 16 \end{pmatrix}.$$

(b) The phase diagram for the associated homogeneous system is



Now, we translate this picture to the equilibrium point: