Math 2550, Exam 3 PRACTICE

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1. Compute the integrals

   (a) (10 points) (15.1.24)

$$\int_{[0,1]\times[0,1]} \frac{y}{x^2y^2+1}\, dA$$

   (b) (10 points) (15.2.50)

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dydx$$

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Solution:

(a)

$$
\begin{aligned}
\int_0^1 \left(\int_0^1 \frac{y}{x^2y^2+1}\, dy \right) dx &= \int_0^1 \frac{1}{2x^2} \left(\int_1^{x^2+1} \frac{1}{u}\, du \right) dx \\
&= \int_0^1 \frac{1}{2x^2} \left(\ln|x^2+1| \right) dx \\
&= \int_0^1 \frac{1}{2x^2} \left(\ln(x^2+1) \right) dx \\
&= \frac{1}{2}\left[-\frac{1}{x}\ln(x^2+1)\Big|_{x=0}^{1} - \int_0^1 \frac{2}{x^2+1}\, dx \right] \\
&= \frac{1}{2}\left[-\ln(2) - 2\tan^{-1} x\Big|_{x=0}^{1} \right] \\
&= -\frac{\ln(2)}{2} - \tan^{-1}(1) \\
&= -\frac{\ln(2)}{2} - \frac{\pi}{4}.
\end{aligned}
$$

(b)

$$
\begin{aligned}
\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy dx &= \int_0^4 \left(\int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx \right) dy \\
&= \int_0^4 \frac{e^{2y}}{4-y} \left(\int_0^{\sqrt{4-y}} x\, dx \right) dy \\
&= \int_0^4 \frac{e^{2y}}{4-y} \left(\frac{4-y}{2} \right) dy \\
&= \frac{1}{2} \int_0^4 e^{2y}\, dy \\
&= \frac{e^{2y}}{4}\Big|_{y=0}^{4} \\
&= (e^8 - 1)/4.
\end{aligned}
$$

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2. (15.4.22) Consider the integral

$$\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\, dydx.$$

(a) (10 points) Describe (or draw) the region of integration.

(b) (10 points) Change variables to polar coordinates and evaluate the integral.

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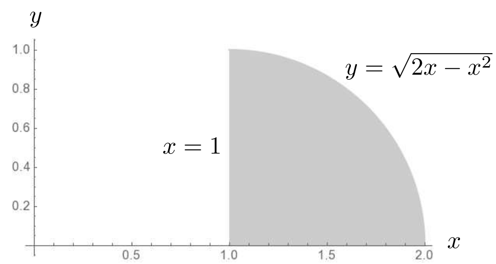
Solution:

(a) Considering the upper limit of the inner integral, we have a boundary $y = \sqrt{2x - x^2}$. This is the equation of part of a circle:

$$x^2 - 2x + 1 + y^2 = 1$$

with center $(1, 0)$ and unit radius. The region is, therefore, a quarter unit disk.



(b) To figure out the upper limit of integration for the radial variable, we need to find r so that

$$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1.$$

That is, $r^2 - 2r\cos\theta = 0$ or $r = 2\cos\theta$.

$$\begin{aligned}
\int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4}\, r\, dr d\theta &= \int_0^{\pi/4}\left[-\frac{1}{2r^2}\right]_{r=\sec\theta}^{2\cos\theta} d\theta \\
&= -\frac{1}{2}\int_0^{\pi/4}\left[\frac{1}{4\cos^2\theta} - \cos^2\theta\right] d\theta \\
&= \frac{1}{2}\int_0^{\pi/4}\cos^2\theta\, d\theta - \frac{1}{8}\int_0^{\pi/4}\sec^2\theta\, d\theta \\
&= \frac{1}{4}\int_0^{\pi/4}[1 + \cos(2\theta)]\, d\theta - \frac{1}{8}\tan\theta\Big|_0^{\pi/4} \\
&= \frac{\pi}{16} + \frac{1}{8}\sin(2\theta)\Big|_0^{\pi/4} - \frac{1}{8} \\
&= \frac{\pi}{16}.
\end{aligned}$$

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3. (15.5.44) Consider the iterated integral

$$\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin(2z)}{4-z}\, dydzdx.$$

Change the order of integration to write the integral in **five** different ways. Evaluate one of the integrals you obtain.

(a) (4 points) (first change of order)

(b) (4 points) (second change of order)

(c) (4 points) (third change of order)

(d) (4 points) (fourth change of order)

(e) (4 points) (fifth change of order)

(f) (5 points) Evaluation:

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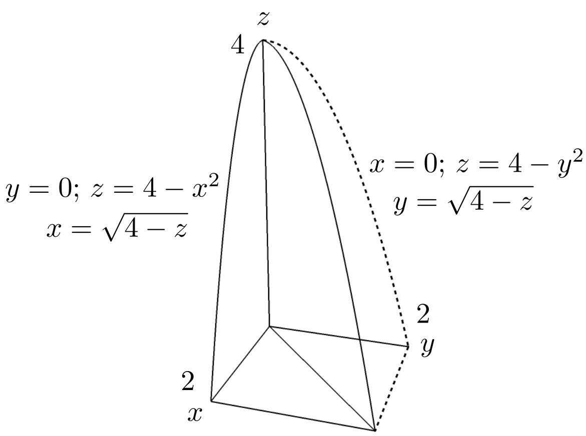
Solution: There are six (3!) ways to write this integral depending on which length elements appear first second and third. One of them is above ($dydzdx$). We can organize the remaining five by starting with the other one that ends with dx followed by the ones that end with dy. It's helpful to draw the volume of integration which is the region under the parabolic cylinder $z = 4 - x^2$ and also in the wedge between the x, z-plane and the plane $y = x$:



Some of the relevant bounding relations are indicated.

(a)

$$\int_0^2 \int_0^x \int_0^{4-x^2} \frac{\sin(2z)}{4-z}\, dzdydx = \int_0^2 \int_0^x \int_0^{4-y^2} \frac{\sin(2z)}{4-z}\, dzdydx.$$

(Since $y = x$ on the inner boundary, there are two ways to write this one.)

(b)

$$\int_0^2 \int_0^{4-y^2} \int_y^{\sqrt{4-z}} \frac{\sin(2z)}{4-z}\, dxdzdy.$$

(c)

$$\int_0^2 \int_y^2 \int_0^{4-x^2} \frac{\sin(2z)}{4-z}\, dzdxdy.$$

(d)

$$\int_0^4 \int_0^{\sqrt{4-z}} \int_y^{\sqrt{4-z}} \frac{\sin(2z)}{4-z}\, dxdydz.$$

(e)

$$\int_0^4 \int_0^{\sqrt{4-z}} \int_0^x \frac{\sin(2z)}{4-z}\, dydxdz.$$

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(f) The problem is that $4-z$ in the denominator of the integrand...and it needs to get canceled by a previous integration. Checking the first few, you see this doesn't happen, so the best bet is to wait for the z integration until the end. This leaves the last two as possibilities. I guess those two are both viable:

$$\begin{aligned}\int_0^4\int_0^{\sqrt{4-z}}\int_y^{\sqrt{4-z}}\frac{\sin(2z)}{4-z}\,dxdydz &= \int_0^4\frac{\sin(2z)}{4-z}\int_0^{\sqrt{4-z}}[\sqrt{4-z}-y]\,dydz\\ &= \int_0^4\frac{\sin(2z)}{4-z}\left[4-z-\frac{4-z}{2}\right]\,dz\\ &= \frac{1}{2}\left[-\frac{\cos(2z)}{2}\right]_{z=0}^4\\ &= \frac{1-\cos 8}{4}.\end{aligned}$$

$$\begin{aligned}\int_0^4\int_0^{\sqrt{4-z}}\int_0^x\frac{\sin(2z)}{4-z}\,dydxdz &= \int_0^4\frac{\sin(2z)}{4-z}\int_0^{\sqrt{4-z}}x\,dxdz\\ &= \int_0^4\frac{\sin(2z)}{2}dz\\ &= \frac{1-\cos 8}{4}.\end{aligned}$$

The last one is clearly the easiest.

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4. (20 points) (15.6.16) A plate is modeled by the region

$$\mathcal{U} = \{(0, y, z) \in \mathbb{R}^3 : y^2 < z < 1\}$$

along with an areal density $\rho = z + 1$ in the given coordinates. If one models uniform rotation of this plate about the z-axis with angular velocity $\omega = 5$ radians per second, calculate the kinetic energy of rotation.

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Solution: The kinetic energy is $I\omega^2/2 = 25I/2$ where I is the moment of inertia

$$\begin{aligned}
I &= \int_{\mathcal{U}} \rho r^2 \\
&= \int_{-1}^{1} \int_{y^2}^{1} (z+1) y^2 \, dz dy \\
&= \int_{-1}^{1} y^2 \left[\frac{1}{2} z^2 + z \right]_{z=y^2}^{1} dy \\
&= \int_{-1}^{1} y^2 \left[\frac{1}{2} + 1 - \left(\frac{1}{2} y^4 + y^2 \right) \right] dy \\
&= \int_{-1}^{1} \left[\frac{3}{2} y^2 - \frac{1}{2} y^6 - y^4 \right] dy \\
&= 2 \left[\frac{1}{2} y^3 - \frac{1}{14} y^7 - \frac{1}{5} y^5 \right]_{y=1} \\
&= \frac{2(35 - 5 - 14)}{70} \\
&= \frac{16}{35}.
\end{aligned}$$

Thus, the kinetic energy is

$$\text{K.E.} = \frac{(16)(25)}{(2)(35)} = \frac{40}{7}.$$

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5. (20 points) (15.7.59) Change variables to **spherical coordinates** to evaluate the iterated integral

$$\int_{\mathcal{U}} \int_{1-\sqrt{1-(x^2+y^2)}}^{\sqrt{x^2+y^2}} 1 \, dz \, dx dy$$

where $\mathcal{U} = \{(x, y, 0) : x^2 + y^2 < 1\}$ is the unit disk in the x, y-plane.

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Solution: This is the volume of the region above the bottom half of the sphere $x^2 + y^2 + (z-1)^2 = 1$ and the cone $z^2 = x^2 + y^2$. The scaling factor for spherical coordinates is $\sigma = \rho^2 \sin\phi$.

$$\int_{\mathcal{U}} \int_{1-\sqrt{1-(x^2+y^2)}}^{\sqrt{x^2+y^2}} 1\, dz\, dxdy = \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^{\rho_{\max}} \sigma\, d\rho d\theta d\phi$$
$$= \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^{\rho_{\max}} \rho^2 \sin\phi\, d\rho d\theta d\phi.$$

To find the upper limit $\rho_{\max}$ for the inner integral (with respect to the radius ρ), we need to use the equation of the sphere and determine $\rho^2 = x^2 + y^2 + z^2$ in terms of ϕ and θ. From the equation in rectangular coordinates we see $\rho^2 = x^2 + y^2 + z^2 = 2z = 2\cos\phi$.

$$\int_{\mathcal{U}} \int_{1-\sqrt{1-(x^2+y^2)}}^{\sqrt{x^2+y^2}} 1\, dz\, dxdy = \int_{\pi/4}^{\pi/2} \sin\phi \int_0^{2\pi} \int_0^{2\cos\phi} \rho^2\, d\rho d\theta d\phi$$
$$= 2\pi \int_{\pi/4}^{\pi/2} \sin\phi \left[\frac{1}{3}\rho^3\right]_{\rho=0}^{2\cos\phi} d\phi$$
$$= \frac{16\pi}{3} \int_{\pi/4}^{\pi/2} \sin\phi \cos^3\phi\, d\phi$$
$$= \frac{16\pi}{3} \left[-\frac{1}{4}\cos^4\phi\right]_{\phi=\pi/4}^{\pi/2}$$
$$= \left(\frac{16\pi}{3}\left(\left[\frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^4\right]\right.\right.$$
$$= \frac{\pi}{3}.$$