**Math 2550, Exam 1 (Practice)** Name and section: ____________________

1. (20 points) (12.3.16) Two rods are arranged end to end. We choose coordinates so that the first rod has an end $A$ at the origin, with the other end $B$ in the first quadrant of the $x, y$-plane, and makes an angle of $\pi/6$ with the positive $x$-axis. The second rod has an end at $B = (b_1, b_2, b_3)$, lies in a plane parallel to the $y, z$-plane, and makes an angle of $\pi/4$ with the horizontal plane through $B$ in such a way that the other end $C = (c_1, c_2, c_3)$ satisfies $c_2 > b_2$ and $c_3 > b_3$. Find the angle between the two rods.

**Solution:**

The coordinates of the point $B$ are $|B|(\sqrt{3}/2, 1/2, 0)$. Since we want the direction of the first rod with respect to the point $B$ where the rods meet, we should take the displacement $(0, 0, 0) - B = -|B|(\sqrt{3}/2, 1/2, 0)$ as the vector representing the direction of the first rod. The coordinates of point $C$ are $|B|(\sqrt{3}/2, 1/2, 0) + |C|(0, \sqrt{2}/2, \sqrt{2}/2)$. Thus, we may take the (displacement) vector $|C|(0, \sqrt{2}/2, \sqrt{2}/2)$ to represent the direction of the second rod. The angle is determined by

$$\cos\theta = \frac{-|B|(\sqrt{3}/2, 1/2, 0) \cdot |C|(0, \sqrt{2}/2, \sqrt{2}/2)}{|B||C|} = -\frac{\sqrt{2}}{4}.$$

Therefore,

$$\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{4}\right).$$

This can also be written as

$$\theta = \pi - \cos^{-1}\left(\frac{\sqrt{2}}{4}\right),$$

and it is about 111°.

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2. (20 points) (12.4.16)

   Find the area of the triangle with vertices $(1, 1, 1)$, $(2, 1, 3)$, and $(3, -1, 1)$.

**Solution:** This is half the area of the parallelogram determined by the vectors

$$(2, 1, 3) - (1, 1, 1) = (1, 0, 2) \qquad \text{and} \qquad (3, -1, 1) - (1, 1, 1) = (2, -2, 0).$$

The area of the parallelogram is given by the norm of the cross product

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

Thus, the area of the triangle is

$$\sqrt{4 + 4 + 1} = 3.$$

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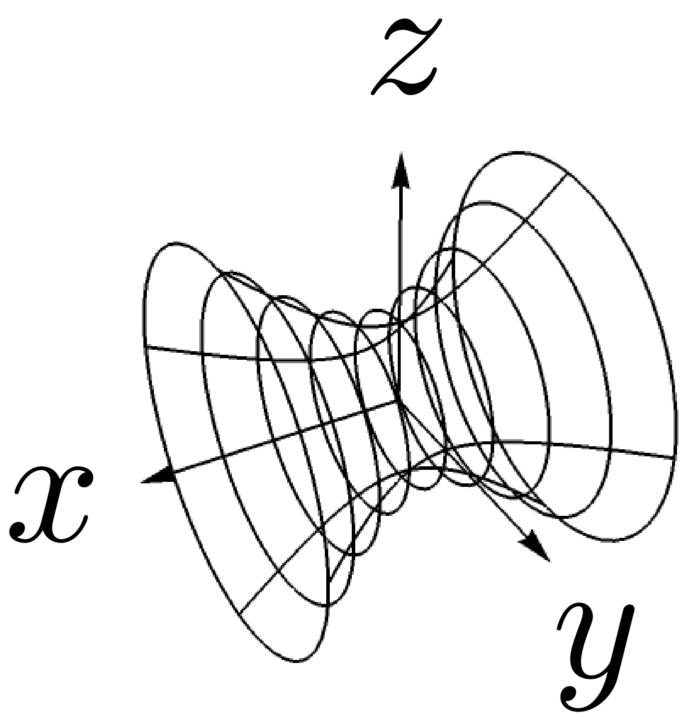
3. (20 points) (12.6.28) Sketch the quadric surface in $\mathbb{R}^3$ associated with the relation

$$y^2 + z^2 - x^2 = 1.$$

**Solution:**



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4. (20 points) (13.2.14) Solve the initial value problem for the vector valued function **r**:

$$\begin{cases} \mathbf{r}' = (t^3 + 4t, t, 2t^2) \\ \mathbf{r}(0) = (1, 1, 0). \end{cases}$$

**Solution:**

$$\mathbf{r} = (1,1,0) + \int_0^t (\tau^3 + 4\tau, \tau, 2\tau^2)\, d\tau = (1,1,0) + \left(\frac{1}{4}t^4 + 2t^2, \frac{1}{2}t^2, \frac{2}{3}t^3\right).$$

This can also be written as

$$\mathbf{r} = \left(1 + \frac{1}{4}t^4 + 2t^2, 1 + \frac{1}{2}t^2, \frac{2}{3}t^3\right).$$

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5. (circular motion) A point mass moves along a circlular path parameterized with respect to time by

$$\mathbf{r}(t) = a(\sin\theta, -\cos\theta)$$

where $a > 0$ is constant and $\theta = \theta(t)$.

(a) (10 points) Find the velocity and acceleration on the mass.

(b) (10 points) Let $\mathbf{u}_1 = (\cos\theta, \sin\theta)$ and $\mathbf{u}_2 = (-\sin\theta, \cos\theta)$. Express the acceleration as a linear combination of $\mathbf{u}$ and $\mathbf{v}$

**Solution:**

(a)

$$\mathbf{v} = \mathbf{r}' = a\theta'(\cos\theta, \sin\theta) = a\theta'\mathbf{u}_1.$$

$$\mathbf{a} = \mathbf{r}'' = a\left[\theta''(\cos\theta, \sin\theta) + (\theta')^2(-\sin\theta, \cos\theta)\right].$$

(b)

$$\mathbf{a} = a\theta''\mathbf{u}_1 + a(\theta')^2\mathbf{u}_2.$$