Math 2403, Final Exam (practice) Name and section: ____________________

1. (a) (5 points) Solve the IVP

$$\begin{cases} y' = y^2 \\ y(2) = 1. \end{cases}$$

(b) (5 points) Plot the solution you found in part (a).

(c) (5 points) What is the *interval of definition* of your solution you found in part (a)?

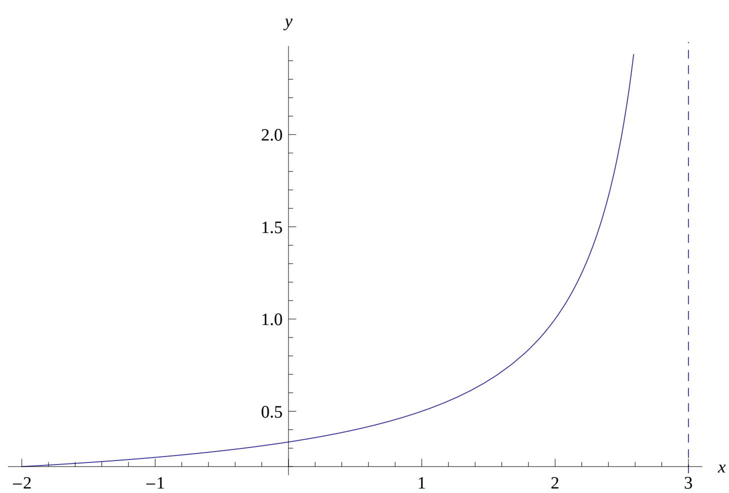
(d) (5 points) Plot the phase line for the ODE $y' = y^2$.

Solution:

(a) Integrating $y'/y^2 = 1$, we find $-1/y + 1 = x - 2$ or

$$y = \frac{1}{3 - x}.$$

(b)



(c) $(-\infty, 3)$.

(d) This is and autonomous ODE with equilibrium at $y = 0$. The phase line is

$$\longrightarrow\longrightarrow\longrightarrow\longrightarrow 0 \longrightarrow\longrightarrow\longrightarrow\longrightarrow \quad y$$

Name and section: ____________________

2. A series circuit contains a 2 Ohm resistor, a 1/48 Fahrad capacitor, a 0.02 Henry inductor, and an adjustable power source.

   (a) (10 points) If the initial charge on the capacitor is 1/16 Coulomb and there is initially no current flowing in the circuit when the power source is switched on to 9 volts, what is the subsequent charge on the capacitor?

   (b) (10 points) Does this physical system constitute an oscillator? Explain.

Solution:

(a) The equation for the charge $q = q(t)$ is

$$0.02q'' + 2q' + 48q = 9.$$

We consider this equation with the initial conditions $q(0) = 1/16$ and $q'(0) = 0$. The characteristic equation $0.02\alpha^2 + 2\alpha + 48 = 0$ has solutions $\alpha = (-2 \pm \sqrt{4 - 3.84})/0.94 = (-2 \pm \sqrt{0.16})/0.04 = -50 \pm 1$. We notice that there are two negative characteristic values, so the solution of the homogeneous equation is

$$q_h = ae^{-51t} + be^{-49t}.$$

A particular solution is given by the constant $q_p = 9/48 = 3/16$. Thus, $q = ae^{-51t} + be^{-49t} + 3/16$ and we proceed to determine the coefficients a and b to satisfy the initial conditions.

$$\begin{cases} a + b = -1/8 \\ 51a + 49b = 0. \end{cases}$$

That is, $a = (-49/8)/(49 - 51) = 49/16$ and $b = (51/8)/(-2) = -51/16$. Thus,

$$q(t) = \frac{49e^{-51t} - 51e^{-49t} + 3}{16}.$$

(b) Since the coefficients in the operator

$$L[q] = 0.02q'' + 2q' + 48q$$

are all positive, we are modelling this system as a (overdamped) damped oscillator which is being forced with a constant force.

Name and section: ____________________

3. Consider the system of ODEs

$$y' = y(5 - y + z)$$
$$z' = -z(5 - z + y)^2.$$

(a) (10 points) Linearize at $y_* = z_* = 0$, and draw the phase diagram for the *linerized* system.

(b) (10 points) A solution of the original nonlinear system starts with $y(0) = 0.1$ and $z(0) = 0$. Determine the limit

$$\lim_{t \nearrow \infty} y(t).$$

Solution:

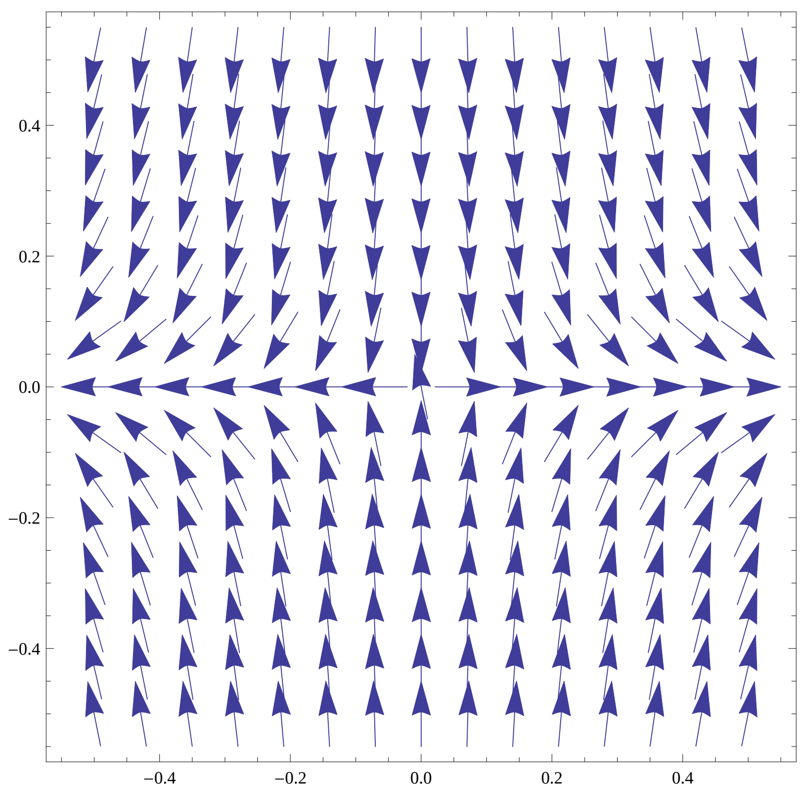
(a) The Jacobian of the vector field $F = (y(5 - y + z), -z(5 - z + y)^2)^T$ is given by

$$DF = \begin{pmatrix} 5 - 2y + z & y \\ -2z(5 - z + y) & -(5 - z + y)^2 + 2z(5 - z + y) \end{pmatrix}.$$

Thus the linearized system at the origin is

$$\mathbf{x}' = \begin{pmatrix} 5 & 0 \\ 0 & -25 \end{pmatrix} \mathbf{x}.$$

The eigenvalues are clearly $\lambda = 5$ (exponential growth in the direction $\mathbf{e}_1$) and $\lambda = -25$ (exponential decay in the direction $\mathbf{e}_2$). Thus, we have a saddle point:



Name and section: ____________________

(b) If we start with $z = 0$, then the second equation indicates $z(t) = 0$ for all t by the existence and uniqueness theorem. Thus, the first equation becomes logistic for y:

$$y' = y(5 - y).$$

Since the carrying capacity $y_* = 5$ is an attractive equilibrium for this autonomous system, we see that

$$\lim_{t \nearrow \infty} y(t) = 5.$$

Name and section: ____________________

4. (20 points) Solve the initial value problem

$$\begin{cases} y'' + 4y = \cos 3t \\ y(0) = 0 = y'(0) \end{cases}$$

determine the *period* of the beats.

Solution:

$$y_h = a\cos 2t + b\sin 2t.$$

$$y_p = A\cos 3t + B\sin 3t.$$

$$L[y_p] = -5A\cos 3t - 5B\sin 3t = \cos 3t.$$

Thus,

$$y(t) = a\cos 2t + b\sin 2t - \frac{1}{5}\cos 3t$$

for some constants a and b.

$$y(0) = a - 1/5 = 0 \quad \text{and} \quad y'(0) = 2b = 0.$$

Therefore,

$$y(t) = \frac{1}{5}(\cos 2t - \cos 3t).$$

Setting

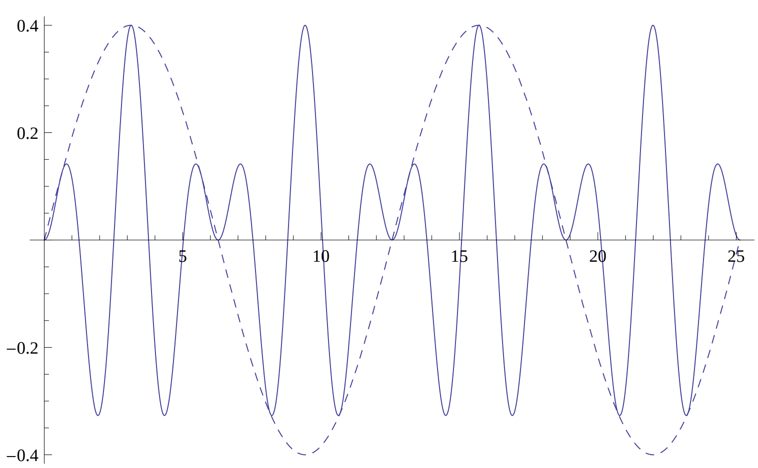
$$\theta = \frac{2t + 3t}{2} \quad \text{and} \quad \phi = \frac{2t - 3t}{2}$$

and using the cosine addition formula, we find

$$y(t) = \frac{2}{5}\sin\left(\frac{5t}{2}\right)\sin\left(\frac{t}{2}\right).$$

The beats are determined by the sine function of lower frequency. That would be frequency $1/2$. And the period is therefore

$$\frac{2\pi}{1/2} = 4\pi.$$



Name and section: ____________________

5. (20 points) Solve the initial value problem

$$\begin{cases} y'' + 2y' - y = t \\ y(0) = 0 = y'(0) \end{cases}$$

by the method of Laplace transforms.

Solution: Consulting the table of Laplace transforms, we find $\mathcal{L}[t] = 1/s$. Thus, given the homogeneous boundary values of the problem, the Laplace transform of the initial value problem is

$$s^2Y + 2sY - Y = \frac{1}{s^2}.$$

Therefore,

$$Y = \frac{1}{s^2(s+1+\sqrt{2})(s+1-\sqrt{2})}.$$

By partial fractions, we find

$$Y = -\frac{2s+1}{s^2} + \frac{2s+5}{s^2+2s-1} = -\frac{2}{s} - \frac{1}{s^2} + \frac{4-3\sqrt{2}}{4(s+1+\sqrt{2})} + \frac{4+3\sqrt{2}}{4(s+1-\sqrt{2})}.$$

Consulting the table (shifting in s), we find

$$Y = -\frac{2}{s} - \frac{1}{s^2} + \frac{4-3\sqrt{2}}{4}\mathcal{L}\left[e^{-(1+\sqrt{2})t}\right] + \frac{4+3\sqrt{2}}{4}\mathcal{L}\left[e^{-(1-\sqrt{2})t}\right].$$

Therefore,

$$y = -2 - t + \frac{4-3\sqrt{2}}{4}e^{-(1+\sqrt{2})t} + \frac{4+3\sqrt{2}}{4}e^{-(1-\sqrt{2})t}.$$