Name: ______________________________

Math 2403: ODE — Final Exam (practice): — July 18, 2011

First Order Equations and Systems, Second Order Equations, Laplace Transform Methods

No calculators, notes, or other aids are allowed. You may use pen or pencil, scratch paper and what you know. Write clearly and neatly; no credit will be given for illegible or messy answers. In this exam y will be a function dependent on either x or t.

1. Solve the equation $y' + 3y = t + e^{-2t}$.

2. Solve the IVP $ty' + (t+1)y = t$, $y(\ln 2) = 1$ for $t > 0$.

3. The velocity v of an object moving away from the earth and starting with positive initial velocity v_0 is described by the equation

$$v\frac{dv}{dy} = -\frac{gR^2}{(R+y)^2}$$

where g is the gravitational constant, R is the radius of the earth, and y is the height of the object above the earth's surface. Assuming the object eventually reaches zero velocity and starts falling back to the earth, find the maximum height. Find the initial velocity (in terms of g and R) at which there is no maximum height.

4. Sketch the phase plane diagram of the linear system

$$\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x}.$$

Classify the equilibrium point as stable, asymptotically stable, or unstable.

5. Sketch the phase plane diagram for the competing species system

$$\begin{cases} x' = x(1 - x - y) \\ y' = y(3/4 - y - x/2) \end{cases}$$

6. The nonlinear system

$$\begin{cases} x' = -(2 + y)(x + y) \\ y' = -y(1 - x) \end{cases}$$

has an equilibrium point which is a saddle. Find that equilibrium and accurately sketch the phase plane diagram of the linearized system at that point. Show your work and clearly indicate the expanding and contracting directions.

7. Find the general solution of the ODE $y'' + 2y' = 3 + 4\sin(2t)$.

8. Find an initial value problem for which the associated algebraic equation (via Laplace Transform) is

$$(s^2 + 2s + 2)Y = \frac{s^2 + 3}{s^2 + 2s + 2}.$$

9. Solve the IVP

$$\begin{cases} y'' + y = \sum_{j=0}^{\infty} (-1)^j h(t - j\pi) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

where

$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

10. Solve the IVP

$$\begin{cases} y'' + y = h(t - 3\pi) \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

11. Consider the harmonic oscillator described by the IVP

$$\begin{cases} y'' + y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Find all positive times T for which an impulsive forcing can be imposed to achieve the condition $y(t) \equiv 0$ for $t \geq T$. With each such time T, describe also the impulsive force.