Power Series in Two Variables

MATH 1502 Calculus II Notes

September 3, 2008

Most of what we have covered so far on sequences and series is probably a review for most of you, so before we leave the topic altogether, I thought it would be nice to cover something that will be new to most, if not all, of you. In fact, what we will cover on power series in two variables is not in your text SH&E, and that is why I have composed these notes for you. Sometimes, you may have to compose such notes for yourself from the lecture, but as that kind of coursework might be new to many of you, I'll give you a little help to get started.

Some basic prerequisites for the topic at hand are in the book, and I will mention some sections to read and problems to work as we go. Even before you read further here, you should go

read §15.1 in S,H,&E. and work exercises 15.1.2, 6, and 31.

Real valued functions of more than one variable are basic objects in mathematical analysis. The first example in Chapter 15 (S,H,&E) is

$$f(x, y) = xy.$$

In this case, $f : \mathbb{R}^2 \to \mathbb{R}$.

Exercise

1. Graph the function $f(x, y) = xy$ using x, y, z-coordinate axes in 3-D space. Hint: Consider the subsets of $\mathbb{R}^2$ where $x = 0$, where $y = 0$, and where $x = y$.

We will restrict our attention to two (independent) variables in these notes and describe a natural version of Taylor series that works for these functions. These series also have a *center* where they always converge to the function value. Let the center be $(x_0, y_0) \in \mathbb{R}^2$. Then the series has the basic (polynomial) form

$$\sum a_{ij}(x - x_0)^i (y - y_0)^j$$

where the indices i and j run (independently) from 0 to ∞. For example the terms up to order 2 are

$$\begin{aligned} a_{00} + & a_{10}(x - x_0) + a_{01}(y - y_0) \\ & + a_{20}(x - x_0)^2 + a_{11}(x - x_0)(y - y_0) + a_{02}(y - y_0)^2. \end{aligned} \tag{1}$$

This is the order 2 Taylor polynomial $P_2(x, y)$. In Calculus III, one learns to understand the behavior of all such "quadratic polynomials" in two variables.

Exercises

2. Write down the general form of the order 3 terms in the Taylor series. How many order 4 terms are there?

3. Rewrite the order 2 Taylor polynomial (given in (1)) in the form

   $$P_2(x, y) = \sum_{i=0}^{2} q_i(y)(x - x_0)^i$$

   and in the form

   $$P_2(x, y) = \sum_{j=0}^{2} q_j(x)(y - y_0)^j.$$

In order to make our recipe for writing down a Taylor series complete, we need to explain how to get the coefficients a_{ij} in terms of the function f. As before, the zero order (constant) term is just $a_{00} = f(x_0, y_0)$ and the other terms involve derivatives of the function f evaluated at (x_0, y_0) and factorials. After the first few, the pattern will be clear. Let us use as an example the function $f(x, y) = \tan^{-1}(y/x)$ at the center point $(x_0, y_0) = (1, 1)$.

To get a_{10}, let us break the procedure up into two steps:

1. treat y as a constant, and differentiate with respect to x. This is called "taking the partial derivative w.r.t. x," and if you wish to read more about it, you can look at S,H,&E §15.4.

2. Plug the center point (x_0, y_0) into the resulting expression.

The expression (partial derivative w.r.t. x) you find in the first step is denoted by f_x or $\frac{\partial f}{\partial x}$ and may also simply be called the "first x-partial," the "first homogeneous x-partial," or simply the "x-partial." For example if $f(x, y) = \tan^{-1}(y/x)$, then the x-partial is given by

$$\frac{\partial f}{\partial x} = \frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}.$$

In step 2, we evaluate this expression at $(x_0, y_0) = (1, 1)$ to obtain

$$a_{10} = -\frac{1}{2} = \frac{-1/2}{1!\ 0!}$$

Thus, the coefficient of the first order linear term in x is $a_{10} = -1/2$.

To get a_{01}, first treat x as a constant and differentiate w.r.t. y; i.e., take the y-partial. Second plug in (x_0, y_0).

Exercise

4. Compute a_{01} for $f(x, y) = \tan^{-1}(y/x)$ at $(x_0, y_0) = (1, 1)$.

Solution:

$$\frac{\partial f}{\partial y} = \frac{1/x}{1 + (y/x)^2} = \frac{x}{x^2 + y^2} \to \frac{1}{2}.$$

$$a_{01} = \frac{1}{2} = \frac{1/2}{0!1!}. \quad \square$$

In order to get the 2nd order coefficients, we take more derivatives and introduce nontrivial factorials. Here is how to compute a_{20} in three steps:

1. Differentiate twice w.r.t. x.

2. Plug in $(x_0.y_0)$.

3. Divide by $2!0! = 2$. (Note: This is a product of factorials $i!j!$. The i is the power of $x - x_0$ in this term (which is the same number as the number of x-partials you are taking), and the j is the power of $y - y_0$ in this term (or the number of y-partials you are taking).

The second partial derivative computed in step 1 (which is a function) is denoted by f_{xx} or

$$\frac{\partial^2 f}{\partial x^2}.$$

It is the *second order homogeneous x-partial.* After evaluation (plugging in), we can denote the resulting number by

$$f_{xx}(x_0, y_0) \qquad \text{or} \qquad \frac{\partial^2 f}{\partial x^2}\,(x_0, y_0).$$

Thus after completing the three steps, we find

$$a_{20} = \frac{1}{2!0!}\,\frac{\partial^2 f}{\partial x^2}\,(x_0, y_0),$$

And for our example $f(x, y) = \tan^{-1}(y/x)$,

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) \\ &= \frac{2xy}{(x^2+y^2)^2} \\ \frac{\partial^2 f(1,1)}{\partial x^2} &= \frac{1}{2}\,. \end{aligned}$$

The coefficient of the xy-term, a_{11}, is calculated by taking one derivative w.r.t. x, then one w.r.t. y, plugging in (x_0, y_0) and dividing by $1!1!$. This second partial is denoted f_{xy} or $\partial^2 f/\partial y \partial x$ and

$$a_{11} = f_{xy}(x_0, y_0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\Big|_{(x_0,y_0)} = \frac{\partial^2 f}{\partial y \partial x}\,(x_0, y_0).$$

Note: The derivative appearing here is a *second order mixed partial.* It is not homogeneous because we are differentiating with respect to both x and y; it usually doesn't matter which partial you compute first (see S,H,&E §15.6 for more details).

For example, with $f(x, y) = \tan^{-1}(y/x)$,

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) \\ &= -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} \\ &= \frac{y^2-x^2}{x^2+y^2} \end{aligned}$$

and

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1}{x^2+y^2} - \frac{2x^2}{x^2+y^2}$$
$$= \frac{y^2-x^2}{x^2+y^2}.$$

You should be able to see the pattern by now.

Exercises

5. Compute the coefficient of $(y-1)^2$

$$a_{02} = \frac{1}{0!2!}\frac{\partial^2 f}{\partial y^2}(1,1)$$

when $f(x,y) = \tan^{-1}(y/x)$. What is the full second order Taylor polynomial in this case?

6. Find the second order Taylor expansion of $f(x,y) = x^2\cos y + y^2 \sin x$ at $(x_0, y_0) = (0,0)$.

7. Extra 15.6.4, 15.4.57

The general formula is given by

$$a_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x_0, y_0)$$

where $\frac{\partial^{i+j} f}{\partial x^i \partial y^j}$ means to differentiate i times w.r.t. x and then j times w.r.t. y. (The order in which you compute these partial derivatives usually doesn't matter.)

Exercise

8. Compute the Taylor Series for $f(x,y) = \ln(xy)$ at $(x_0,y_0) = (1,1)$.

Adding, Scaling, and Multiplying Vectors

If (a, b) and (c, d) are points in $\mathbb{R}^2$, i.e., vectors, then we can *add* them componentwise:

$$(a, b) + (c, d) = (a + c, b + d).$$

It is also possible to *scale* a vector by a constant $\alpha \in \mathbb{R}$:

$$\alpha(x, y) = (\alpha x, \alpha y).$$

The *dot product* of two vectors (§13.3) is defined by

$$(a, b) \cdot (c, d) = ac + bd.$$

Notice that $(a, b), (c, d) \in \mathbb{R}^2$, but $(a, b) \cdot (c, d) \in \mathbb{R}$.

For a function f of two variables x and y, define the (full) *derivative* of f to be the vector Df whose entries are the two first partials of f. That is $Df = (fx, fy)$.

In the exercise below, we use this definition and also denote the point/vector (x, y) by $\mathbf{x}$ and the point (x_0, y_0) by $\mathbf{x}_0$

Exercise

9. Show that the first order Taylor polynomial is given by

$$P_1(\mathbf{x}) = f(\mathbf{x}_0) + Df(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0).$$

Compare this to the Taylor polynomial $P_1(x)$ for a function of one variable.

Matrices; Multiplying Matrices and Vectors

A 2×2 *matrix* is an array of four numbers:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If M is the matrix given above and $\mathbf{x}$ is the vector (x, y), which we will write as a column vector

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix},$$

then

$$M\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Note that $M\mathbf{x} \in \mathbb{R}^2$. At present, we are ignoring the distinction between row vectors (x, y) and column vectors $\begin{pmatrix} x \\ y \end{pmatrix}$; sometimes one pays attention to the difference, but we can easily see what is going on now. An example of a matrix is given by the *Hessian* which is the array of second partials:

$$D^2 f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

The entries f_{xx} and f_{yy} are called the *diagonal* entries. Notice that the off diagonal entries should be f_{xy} and f_{yx}. (Can you explain?)

Exercise

10. Show that

$$P_2(\mathbf{x}) = f(\mathbf{x}_0) + Df(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2} D^2 f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0).$$

Compare this to $P_2(x)$.