Invariant Subspaces and Eigenspaces

MATH 1502 Calculus II Notes

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We want to understand all linear transformations $\mathbb{L} : \mathbb{R}^n \to \mathbb{R}^m$. You should now have some of the basic concepts and terminology of vector spaces at your disposal, and we can address more directly the structure of linear transformations. As we will see, the basic case is the one in which $n = m$. That is to say, the case in which the domain and the target are the same vector space. This special case makes possible the key notion of an *invariant subspace*.

Here are a few high points for these notes:

1. *When a linear transformation* $\mathbb{L} : \mathbb{R}^n \to \mathbb{R}^n$, *it makes sense to talk about an* invariant subspace *of* $\mathbb{R}^n$.
2. *A subspace* $\mathcal{V}$ *of* $\mathbb{R}^n$ *is* invariant *if* $\mathbb{L}(\mathbf{v}) \in \mathcal{V}$ *for every* $\mathbf{v} \in \mathcal{V}$. *In other words* $\mathbb{L}(\mathcal{V}) \subset \mathcal{V}$.
3. *The simplest such situation is that in which the invariant subspace is one-dimensional, i.e., spanned by a single nonzero vector* $\mathbf{v}$. *In this case, the subspace* $\operatorname{span}\{\mathbf{v}\}$ *is called an* eigenspace.
4. *Given an eigenspace* $\operatorname{span}\{\mathbf{v}\}$, *the image of* $\mathbf{v}$ *must be a multiple of* $\mathbf{v}$, *that is,* $\mathbb{L}(\mathbf{v}) = \lambda\mathbf{v}$. *The multiplier* λ *is called the* eigenvalue.

(A Tale of) Two Transformations

Consider $\mathbb{L} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbb{L}(\mathbf{x}) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x}.$$

Notice that the x, y-plane is a subspace of $\mathbb{R}^3$. Furthermore, if we apply $\mathbb{L}$ to a point $(x, y, 0)$ in the x, y-plane, then we get $(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, 0)$ which is again in the x, y-plane. Thus, for this transformation, the x, y-plane is an invariant subspace. (It just gets rotated by the transformation.)

Also, the subspace $\text{span}\{\mathbf{e}_3\}$ is an invariant subspace (an eigenspace) with eigenvalue $\lambda = 3$.

Notice that these two subspaces intersect only in the zero vector. That is, if we call the x, y-plane $\mathcal{A}$, then $\mathcal{A} \cap \text{span}\{\mathbf{e}_3\} = \{\mathbf{0}\}$. An interesting thing happens when two subspaces intersect only in the zero vector.

Exercise

1. Let $\mathcal{A}$ & $\mathcal{W}$ be two subspaces of $\mathbb{R}^n$ with $\mathcal{A} \cap \mathcal{W} = \{\mathbf{0}\}$. Show that
   (a) $\text{span}(\mathcal{A} \cup \mathcal{W}) = \{\mathbf{v} + \mathbf{w} : \mathbf{v} \in \mathcal{A}, \mathbf{w} \in \mathcal{W}\}$.
   (b) For each $\mathbf{x} \in \text{span}(\mathcal{A} \cup \mathcal{W})$, there is a unique $\mathbf{v} \in \mathcal{A}$ and a uniqe $\mathbf{w} \in \mathcal{W}$ such that $\mathbf{x} = \mathbf{v} + \mathbf{w}$.
   (c) If you know the values of a linear transformation $\mathbb{L} : \mathbb{R}^n \to \mathbb{R}^n$ on the subspaces $\mathcal{A}$ & $\mathcal{W}$, then you know its values on $\text{span}(\mathcal{A} \cup \mathcal{W})$.

2. Show that if $\mathbf{w} \in \text{span}\{\mathbf{v}\}$ and $\mathbf{v}$ is an eigenvector with eigenvector λ, then $\mathbf{w}$ is an eigenvector with eigenvalue λ.

3. Show that if $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors with different eigenvalues, then the corresponding eigenspaces intersect only in the zero vector.

4. Draw a picture that illustrates the transformation $\mathbb{L} : \mathbb{R}^3 \to \mathbb{R}^3$ described above.

5. Does the linear transformation of problem 4 have any other invariant subspaces?

Another linear transformation $\tilde{\mathbb{L}} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\tilde{\mathbb{L}}(\mathbf{x}) = \begin{pmatrix} \cos\theta - \sin\theta & -\sin\theta & -\sin\theta \\ 2\sin\theta & \cos\theta + \sin\theta & \cos\theta + \sin\theta - 3 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x}.$$

Exercises

6. Is $\mathbf{e}_3$ an eigenvector for this second transformation? What is the image of $\mathbf{e}_3$? How can you see it from the matrix immediately?
7. Check that $-\mathbf{e}_2 + \mathbf{e}_3$ is an eigenvector for $\tilde{\mathbb{L}}$. What is the eigenvalue?
8. Check that $\text{span}\{\mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2\}$ is an invariant subspace for $\tilde{\mathbb{L}}$. What are $\tilde{\mathbb{L}}(\mathbf{e}_2)$ and $\tilde{\mathbb{L}}(\mathbf{e}_1 - \mathbf{e}_2)$.
9. Can you draw a picture that illustrates the linear transformation $\tilde{\mathbb{L}}$?
10. Does $\tilde{\mathbb{L}}$ have any other invariant subspaces?

Now that we've been using the concept for a while, let us give a formal definition.

Definition: Given a linear transformation $\mathbb{L} : \mathbb{R}^n \to \mathbb{R}^n$, a **nonzero** vector $\mathbf{v}$ is an *eigenvector* for $\mathbb{L}$ if there is some scalar λ for which

$$\mathbb{L}(\mathbf{v}) = \lambda \mathbf{v}.$$

The scalar is called the *eigenvalue*.

From the two transformations above, I hope that you see it can be useful to find invariant subspaces and eigenspaces in particular. Those spaces for the first transformation were almost obvious. At least they should be obvious to you now if you go back and look at the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Looking at the matrix

$$\begin{pmatrix} \cos\theta - \sin\theta & -\sin\theta & -\sin\theta \\ 2\sin\theta & \cos\theta + \sin\theta & \cos\theta + \sin\theta - 3 \\ 0 & 0 & 3 \end{pmatrix},$$

the invariant subspaces of $\tilde{\mathbb{L}}$ are not at all obvious. Nevertheless, I was able to find them (by some process) and tell you what they were. Then you were able to understand the transformation $\tilde{\mathbb{L}}$.

So I hope you see that it would be good to learn methods for finding invariant subspaces and eigenspaces in particular.

Exercises

12. Show that $\mathbf{v}$ is an eigenvector for $\mathbb{L}$ with eigenvector λ if an only if $(A - \lambda I)\mathbf{v} = \mathbf{0}$, i.e., $\mathbf{v} \in ker(A - \lambda I)$, where A is the matrix of $\mathbb{L}$ and I is the identity matrix.

13. Say we have a linear transformation $\mathbb{L}$ with corresponding matrix $A \in M_{n\times n}(\mathbb{R})$. According to the last exercise, we have an equation to find the eigenvalues and eigenvectors:

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

If this is written out, how many scalar equations does it specify and how many scalar unknowns are there?

14. Consider the linear transformation defined by

$$\mathbb{L}(\mathbf{x}) = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \mathbf{x}.$$

Find any invariant subspaces of this transformation.

15. Draw a picture to illustrate the transformation in the last exercise.